Lessons from the Poincaré Group

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Position Operator implies Spin Multiplets

Comment on Wightman Functions and Haag's Theorem

Interaction and Generalized Wave Operators

Relativistic Center Variables

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Position Operators X and Rotational Invariance

Momentum P generates translations of position X (Heisenberg group)

$$e^{iPa} X e^{-iPa} = X + a \ , \ e^{iPa} e^{iXb} e^{-iPa} = e^{i(X+a)b}$$

implies Heisenberg algebra

$$[X^i, X^j] = 0 = [P_i, P_j] \ , \ [X^i, P_j] = i\delta^i{}_j$$

Theorem: If the spatial position X and momentum P transform under rotations as vectors, then the rotations are generated by $M_{ij} = -M_{ji}$

$$M_{ij} = L_{ij} + S_{ij} \ , \ L_{ij} = X^i P_j - X^j P_i \ , \ [X^i, S_{kl}] = 0 = [P_j, S_{kl}]$$

where L_{ij} (orbital angular momentum) and S_{ij} (spin) span two angular momentum algebras which commute with each other.

Proof (Bo-Sture Skagerstam)

By the Heisenberg algebra, L_{ij} generate the same transformations of X and P as M_{ij}. Both realize the Lie algebra of $SO(D-1)$.

$$[M_{ij}, X^k] = [L_{ij}, X^k] = i\big(\delta_{ik}X^j - \delta_{jk}X^i\big)\ ,\ [L_{ij}, P_l] = [M_{ij}, P_l] = i\big(\delta_{il}P_j - \delta_{jl}P_i\big)\ ,$$

$$S_{ij} = M_{ij} - L_{ij}\ ,\ [S_{ij}, X^k] = 0 = [S_{ij}, P_l]\ ,\ [S_{ij}, L_{kl}] = 0\ ,$$

$$if_{ab}{}^{c}M_c = [L_a + S_a, L_b + S_b] = if_{ab}{}^{c}L_c + [S_a, S_b]\ ,\ \text{so } [S_a, S_b] = if_{ab}{}^{c}S_c\ .$$

Consequence

A position operator for massless particles requires $SO(D-1)$ multiplets and not only irreducible helicity multiplets (induced by $SO(D-2)$ multiplets)

$$
\begin{aligned}
\bigl(-iM_{12}\Psi\bigr)_N(p) &= -\bigl(p_x\partial_{p_y} - p_y\partial_{p_x}\bigr)\Psi_N(p) - i\,h\,\Psi_N(p)\ ,\\
\bigl(-iM_{31}\Psi\bigr)_N(p) &= -\bigl(p_z\partial_{p_x} - p_x\partial_{p_z}\bigr)\Psi_N(p) - i\,h\,\frac{p_y}{|\vec{p}| + p_z}\Psi_N(p)\ ,\\
\bigl(-iM_{32}\Psi\bigr)_N(p) &= -\bigl(p_z\partial_{p_y} - p_y\partial_{p_z}\bigr)\Psi_N(p) + i\,h\,\frac{p_x}{|\vec{p}| + p_z}\Psi_N(p)\ ,\\
\bigl(-iM_{01}\Psi\bigr)_N(p) &= |\vec{p}|\partial_{p_x}\Psi_N(p) - i\,h\,\frac{p_y}{|\vec{p}| + p_z}\Psi_N(p)\ ,\\
\bigl(-iM_{02}\Psi\bigr)_N(p) &= |\vec{p}|\partial_{p_y}\Psi_N(p) + i\,h\,\frac{p_x}{|\vec{p}| + p_z}\Psi_N(p)\ ,\\
\bigl(-iM_{03}\Psi\bigr)_N(p) &= |\vec{p}|\partial_{p_z}\Psi_N(p)\ .
\end{aligned}
$$

Elephant in the Room

If a relativistic theory with massless states of maximal helicity h does not contain all helicities $h, h-1, \ldots, -h$ then there cannot exist a position operator X which transforms as a spatial vector.

The theorem shows that it was not personal inability why Wigner could not construct a position operator for massless particles.

The theorem exhibits as inconsistent the spectrum of (covariant and lightcone) string theory which postulate or imply spatial position and momentum operators and massless states which fail to constitute complete spin multiplets.

"Elephant in the room" is an American English metaphorical idiom for an obvious problem that no one wants to discuss.

Mice in the Room

Even complete massless spin multiplets, e.g. $h = 0$ or $h = 1/2, -1/2$, do not allow a spatial position operator

$$\big(-iM_{ij}\Psi\big)(p) = -\big(p^i\partial_{p^j} - p^j\partial_{p^i}\big)\Psi(p) - iS_{ij}\Psi(p)$$
$$\big(-iM_{0i}\Psi\big)(p) = |\vec{p}|\partial_{p^i}\Psi(p) - \frac{p^i}{2|\vec{p}|}\Psi(p) - i\,S_{ij}\frac{p^j}{|\vec{p}|}\Psi(p)$$

Not smooth at $\vec{p} = 0$ violating theorems on generators of Lie groups.

These generators have no invariant domain in common with the Heisenberg group, $\mathcal{S}(S^{D-2} \times \mathbb{R}) \neq \mathcal{S}(\mathbb{R}^{D-1})$.

Whether a mathematical inconsistency is subtle or obvious: all deductions which rely on it are inconclusive. Ex falso aliquid. Mouse $\equiv$ Elephant.

Wightman Distributions and Haag's Theorem

$$W(x_1, \ldots x_n) = \langle \Phi(x_1) \ldots \Phi(x_n) \rangle$$

Theorem assures the reconstruction of fields $\Phi(x)$ and representation $U_{a,\Lambda}$

$$U_{a,\Lambda}\Phi(x)U_{a,\Lambda}^{-1} = D(\Lambda)^{-1}\Phi(\Lambda x + a)$$

unique *up to unitary equivalence*.

Haag's Theorem: If a Hilbert space allows for two sets of local fields, then both are free if one is free.

Popularly: The interaction picture exists only if there is no interaction.

Irrelevant, because the interacting time evolution is unitarily equivalent to the free time evolution.

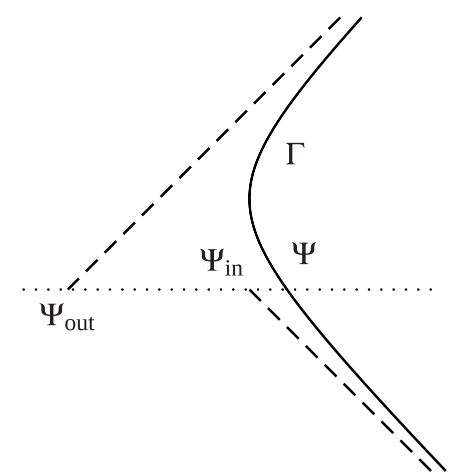
The Wightman distributions do not distinguish between interacting time evolution and free time evolution.

Free and Interacting Time Evolution

The generators of the product representation $U_0 : (a, \Lambda) \to U_{a,\Lambda} \otimes U_{a,\Lambda}$ act by the product rule on two-particles states

$$(P^m \Psi)^{ij}(p_1, p_2) = (p_1^m + p_2^m)\, \Psi^{ij}(p_1, p_2)$$

preserving individual momenta: free time evolution $t \mapsto \Psi(t) = e^{-iP^0 t}\Psi(0)$



Nontrivial motion (one parameter group) has no limit $\Psi_\pm$ otherwise

$\Psi(s + t) = U(s)\Psi(t) \to \Psi_\pm = U(s)\Psi_\pm \to \Psi(t) - \Psi_\pm = U(t)(\Psi(0) - \Psi_\pm)$

$\|\Psi(t) - \Psi_\pm\| = \text{const} \to 0 \Leftrightarrow \Psi(t) = \Psi(0)$

Generalized Wave Operators $\Omega_\pm$

$$\Omega(t) = e^{iH_0 t} e^{-iHt} , \ \lim_{t\to\infty} \Omega(t)\,\Psi = \Omega_-\Psi = \Psi_{\mathsf{out}} , \ \lim_{t\to-\infty} \Omega(t)\,\Psi = \Omega_+\Psi = \Psi_{\mathsf{in}}$$

strong limit (norm limit cannot exist)

S matrix $\quad S = \Omega_+(\Omega_-)^{-1}$

relativistic if $U_0 S = S U_0$

$\Omega_\pm$ intertwine unitarily U (interacting) with $U_0 = \Omega_\pm U \Omega_\pm^{-1}$ (free)

Center Variables

Nonrelativistic center of mass coordinates decompose $H_0 = H_{c.m.} + H_{rel}$

$$\vec{R} = (m_1\vec{x}_1 + m_2\vec{x}_2)/(m_1 + m_2)\ ,\ \vec{r} = \vec{x}_1 - \vec{x}_2$$

relativistic center variables factorize $H_0 = H_{c.m.}(u)H_{rel}(q)$:

$u = (p_1 + p_2)/\sqrt{(p_1 + p_2)^2}\ ,\ q_i = L_u^{-1}(p_i - u(u \cdot p_i)) \in \mathbb{R}^3\ ,\ \sum_i q_i = 0$

$$\begin{aligned} H_0 &= \sqrt{1 + \vec{u}^2}\Big(\sqrt{m_1^2 + \vec{q}^2} + \sqrt{m_2^2 + \vec{q}^2}\Big) \\ &= m_1 + m_2 + \vec{p}^2/(2(m_1 + m_2)) + \vec{q}^2/(2\mu) + \dots \end{aligned}$$

In the center variables relativistic and nonrelativistic scattering and bound state calculations completely analogous $\omega_i(\vec{q}_i) = (\vec{q}_i)^2/(2m_i) \to \sqrt{m_i^2 + \vec{q}_i^2}$

Interaction changes $M^2 = (P_1 + P_2)^2$, $[M^2, M'^2] \neq 0$.

Does it preserve $u = P/M = u'$ (covariant) or $\vec{P} = \vec{P}'$ (canonical, Haag)?

Conclusion

Revising a book on relativistic physics without sweeping ununderstood problems under the carpet is worthwile.

Results are results whether welcome or not.