No Go Theorems for Quantum Fields and Strings

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Haag's Theorem

No Physical States of the Covariant String

Limit $m \to 0$

Lorentz Generators

No Commuting Position Operator

Helicity is a Monopole in Momentum Space

Consolation

Nobody Cares about Haag's Theorem

A Hilbert space, which is generated from free fields acting on the vacuum does not allow for interacting fields.

Each unitary representation of the Poincaré group decomposes *uniquely* into the sum or integral of irreducible representations. These are characterized – irrespective of a coupling g – by their mass and spin s $(m > 0)$ or by their helicity h.

Two particle states: $(P^0\Psi)(p_1, p_2) = (p_1^0 + p_2^0)\,\Psi(p_1, p_2)$

conserves p_1 and p_2 separately, i.e. $H = P^0$ and the product rule imply free motion

Solution: $H_g = M_g Q^0$, $Q = P/\sqrt{P^2}$, $Q^2 = 1$

The massive representations of the *Lorentz* group are independent (!) of M_g, Interaction changes the mass and the size of the momentum of multiparticle states which may form bound states.

In the Heisenberg picture (Ptolemaic system) inconceivable

$$U_t \Phi(t_1, \vec{x}_1)\Phi(t_2, \vec{x}_2)U_t^{-1} \overset{?}{\neq} \Phi(t_1 + t, \vec{x}_1)\Phi(t_2 + t, \vec{x}_2)$$

No Go Theorems of String Theory: $D \in \{3, 10, 26\}$

else a given selection rule fails or given generators have incorrect commutators.

How about better selection rules or better generators?

Does the construction of Lorentz generators succeed completely?

Oversimplification: $\langle P|P\rangle = 1$, $\langle P|P'\rangle = 0$ if $P \neq P'$

Surely, you are joking! (completely discontinuous) , If 5 is even then ...

George Mackey: (if $g \mapsto \langle \Psi | U_g \Phi \rangle$ measurable functions of the Poincaré group)
Representations of the Poincaré group act on wave functions $\Psi : \mathcal{M} \to \mathbb{C}^{2s+1}$

$$(U_a \Psi)(p) = e^{ip\cdot a}\Psi(p) \ , \ (U_\Lambda \Psi)(\Lambda p) = R(W(\Lambda, p))\, \Psi(p)$$

and can be expanded on analytic Schwartz functions ($\mathcal{H}$ = Cauchy completion)

Covariant String: Stone von Neumann, No Physical State

Given d hermitean Heisenberg pairs $[X^m, P_n] = i\delta^m{}_n$, $[X^m, X^n] = 0 = [P_m, P_n]$

$$(P_m\Psi)(p) = p_m\Psi(p)\ ,\ (X^n\Psi)(p) = i\partial_n\Psi(p)\ ,\ \langle\Phi|\Psi\rangle = \int d^dp\, \Phi^*(p)\, \Psi(p)$$

The mass shell condition for physical states $(P^2 - M_i^2)\Psi_i = 0$ implies $\Psi_i = 0$.

Proof: support of Ψ has d-dimensional measure 0.

Impose the mass condition *before* quantisation, X_n not needed for $X_mP_n - X_nP_m$

Quantum Field Theory has no operator X^0 and $P^0 = \sqrt{m^2 + \vec{P}^2}$, i.e. $[\vec{X}, P^0] \neq 0$ (otherwise particles could not move)

Klein Gordon equation describes a *path* in Hilbert space, not a state.

Schrödinger: $i\partial_t\Psi = \sqrt{m^2 + \vec{P}^2}\,\Psi$ (prevents exact localization over finite time)

String Theory Contains $m = 0$, Smooth Limit $m \to 0$?

Trivial: $(\Box + m^2) \to \Box$ is smooth, because $f(m^2) = m^2$ is smooth everywhere

$\frac{1}{p_+} = \frac{1}{p^0+p^3} = \frac{1}{\sqrt{m^2+\vec{p}_T^2+p_z^2}+p_z} \sim|_{\mathcal{A}_3^-} \frac{-p_3}{2(m^2+\vec{p}_T^2)}$, $\quad m = 0 : \frac{-p_3}{2\,\vec{p}_T^2}$ singular on $\mathcal{A}_3^-$

$\mathcal{M} = \{p : p^2 = m^2, p^0 > 0\} \sim \mathbb{R}^3$, $\mathcal{M}_0 = \{p : p^2 = 0, p^0 > 0\} \sim S^2 \times \mathbb{R}$

$p \in \mathcal{M}$: Stability group $SO(3)$, compact, representations unitary, decomposable

$p \in \mathcal{M}_0$: Stability group $E(2)$, not compact, has indecomposable representations

$$\begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} =|_{m=0} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} (n_1^*, n_2^*) \ , \ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{p^1 - ip^2}{\sqrt{p^0 + p^3}} \\ \sqrt{p^0 + p^3} \end{pmatrix}$$

$n_1(p)$ discontinuous on $\mathcal{A}_3^-$, $n_1(p), n_2(p)$ not analytic at $p = 0$, ‘Laurent series’

massive representations of the *Lorentz* group are in terms of $q = p/\sqrt{p^2}$, mass independent, only the spectrum of P depends on m

Massless Generators of the Lorentz Group in the North

$$
\begin{aligned}
\big(-iM_{12}\Psi\big)_N(p) &= -\big(p_x\partial_{p_y} - p_y\partial_{p_x}\big)\Psi_N(p) - i\,h\,\Psi_N(p)\ ,\\
\big(-iM_{31}\Psi\big)_N(p) &= -\big(p_z\partial_{p_x} - p_x\partial_{p_z}\big)\Psi_N(p) - i\,h\,\frac{p_y}{|\vec{p}| + p_z}\Psi_N(p)\ ,\\
\big(-iM_{32}\Psi\big)_N(p) &= -\big(p_z\partial_{p_y} - p_y\partial_{p_z}\big)\Psi_N(p) + i\,h\,\frac{p_x}{|\vec{p}| + p_z}\Psi_N(p)\ ,\\
\big(-iM_{01}\Psi\big)_N(p) &= |\vec{p}|\partial_{p_x}\Psi_N(p) - i\,h\,\frac{p_y}{|\vec{p}| + p_z}\Psi_N(p)\ ,\\
\big(-iM_{02}\Psi\big)_N(p) &= |\vec{p}|\partial_{p_y}\Psi_N(p) + i\,h\,\frac{p_x}{|\vec{p}| + p_z}\Psi_N(p)\ ,\\
\big(-iM_{03}\Psi\big)_N(p) &= |\vec{p}|\partial_{p_z}\Psi_N(p)
\end{aligned}
$$

satisfy the Lorentz algebra for each $h \in \mathbb{R}$, singular on $\mathcal{A}_3^-$, hermitean?

$$
\Psi_S(p) = e^{2i\,h\varphi(p)}\,\Psi_N(p)
$$

only defined for $2h \in \mathbb{Z}$. h is the angular momentum in direction of $\vec{p}$.

Massless Generators of the Lorentz Group in the South

$$\big(-iM_{12}\Psi\big)_S(p) = -\big(p_x\partial_{p_y} - p_y\partial_{p_x}\big)\Psi_S(p) + i\,h\,\Psi_S(p)\ ,$$
$$\big(-iM_{31}\Psi\big)_S(p) = -\big(p_z\partial_{p_x} - p_x\partial_{p_z}\big)\Psi_S(p) - i\,h\,\frac{p_y}{|\vec{p}| - p_z}\Psi_S(p)\ ,$$
$$\big(-iM_{32}\Psi\big)_S(p) = -\big(p_z\partial_{p_y} - p_y\partial_{p_z}\big)\Psi_S(p) + i\,h\,\frac{p_x}{|\vec{p}| - p_z}\Psi_S(p)\ ,$$
$$\big(-iM_{01}\Psi\big)_S(p) = |\vec{p}|\partial_{p_x}\Psi_S(p) + i\,h\,\frac{p_y}{|\vec{p}| - p_z}\Psi_S(p)\ ,$$
$$\big(-iM_{02}\Psi\big)_S(p) = |\vec{p}|\partial_{p_y}\Psi_S(p) - i\,h\,\frac{p_x}{|\vec{p}| - p_z}\Psi_S(p)\ ,$$
$$\big(-iM_{03}\Psi\big)_S(p) = |\vec{p}|\partial_{p_z}\Psi_S(p)\ .$$

Ψ_N analytic in the north, Ψ_S analytic in the south, analytic transition function: analytic section

Topological Obstruction to Commuting Position Operator

$X_i = i\partial_{p^i}$ singular on analytic sections,

covariant derivative (vector under rotations)

$$D_N = \begin{pmatrix} \partial_{p_x} \\ \partial_{p_y} \\ \partial_{p_z} \end{pmatrix}_N - \frac{i\,h}{|\vec{p}|(|\vec{p}| + p_z)} \begin{pmatrix} p_y \\ -p_x \\ 0 \end{pmatrix} \;,\; D_S = \begin{pmatrix} \partial_{p_x} \\ \partial_{p_y} \\ \partial_{p_z} \end{pmatrix}_S + \frac{i\,h}{|\vec{p}|(|\vec{p}| - p_z)} \begin{pmatrix} p_y \\ -p_x \\ 0 \end{pmatrix}$$

Noncommutative Geometry:

$$[D_{p^i}, D_{p^j}] = F_{ij}(p) = ih\epsilon_{ijk}\frac{p^j}{|\vec{p}|^3}$$

$$\mathbb{F} = \frac{1}{2}dx^m\, dx^n\, F_{mn} \;, \qquad \int_{S^2}\mathbb{F} = 4\,\pi\,i\,h$$

independent of the connection $\vec{A}$.

Helicity is a topological charge, a magnetic monopole, in momentum space

No Go Lightcone String

The lightcone string uses Heisenberg pairs of commuting position operators, which however, cannot exist together with a massless unitary representation of the Poincaré group (not even for $h = 0$) in a Hilbert space.

Other deficiencies: excitation operators α^i_{-l} have to be members of a continuous set $U_\Lambda \alpha^i_{-l} U_\Lambda^{-1}$ because they change momentum. Their commutator with α^j_l has to involve a δ-function of the continuous labels rather than a Kronecker-δ.

$[X^-, P^+] = i$ implies negative energy shells.

Einstein’s Consolation of True Believers

Subtle is the Lord

malicious He is not.