Surprises in Relativistic Quantum Physics

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Domains matter

Points matter

Pictures matter

Language matters

Stringtheory inconsistent

Massive very different from Massless

Interaction $P'^n = M'u^n$, $P^n = Mu^n$, $M'_{kl} = M_{kl}$

Axiomatic Quantum Field Theory? Canonical Quantization?
The Wigner Rotation $\mathcal{W}(\Lambda, p) = L_{\Lambda p}^{-1} \Lambda L_p$

Area $\delta$ of an hyperbolic triangle or segment

Calculation a Herculean effort, tedious manipulations

Back of the envelope calculation in SL(2, $\mathbb{C}$)

$$\tan \frac{\delta}{2} = \frac{\sin \varphi}{(\coth \frac{a}{2} \coth \frac{b}{2}) + \cos \varphi}$$

Determines generators of Poincaré group $\mathfrak{P}$ for massive states

$$(P^m \Psi)(p) = p^m \Psi(p) , \quad P^2 = m^2 > 0 ,$$

$$(-iM_{ij} \Psi)(p) = -(p^i \partial_{p^j} - p^j \partial_{p^i}) \Psi(p) + \Gamma_{ij} \Psi(p) ,$$

$$(-iM_{0i} \Psi)(p) = p^0 \partial_{p^i} \Psi(p) + \Gamma_{ij} \frac{p^j}{p^0 + m} \Psi(p)$$
Language and Notation

$\Psi(p)$ rather than $|p_1, s_1 \ldots p_n, s_n\rangle$ allows to check the domain of the generators

Gårding space of smoothened states (smooth $f$ of compact support)

$$\Psi_f = \int_{\mathfrak{p}} d\mu_g f(g) U_g \Psi , \quad U_g \Psi_f = \Psi_{f \circ g^{-1}}$$

requires $g \mapsto U_g \Psi$ measurable $\forall \Psi$ i.e. $U_g$ strongly measurable (a very weak req.)

Introduces differential geometry into Hilbert space

The Gårding space of any strongly measurable unitary representation of a Lie group is dense and invariant under all $U_g$ and the algebra of its generators.

Poincaré algebra acts on Schwartz space $S_M \subset \mathcal{L}^2(M)$, $|p^\alpha \partial_\beta \Psi(p)| < C_{\alpha,\beta,\Psi}$

Sweet poisoned gift
Death of the Covariant String

\[ [P^m, P^n] = 0 = [X_m, X_n], \ [P^n, X_m] = i\delta^m_n, \ e^{iP^a X_m} e^{-iP^a} = X_m - a_m \]

Stone-von Neumann Theorem (1931)

\[ (P^m \psi)(p) = p^m \psi(p), \ (X_n \psi)(p) = -i\frac{\partial}{\partial p^n} \psi(p), \ \langle \psi | \Phi \rangle = \int d^D p \ \psi^*(p) \Phi(p) \]

Algebra acts on \( S^D \), spectrum of \( P^m P^n \eta_{mn} \) continuous

Constraint: \( (P^m P^n \eta_{mn} - M^2(N)) \psi_{\text{phys}} = 0 \)

No nonvanishing smooth solution, not even a measurable solution, support of \( \psi_{\text{phys}} \) has \( D \)-dimensional measure 0 \( \Rightarrow \) \( \psi_{\text{phys}} = 0 \)

The covariant string has no physical states.

\[ 0 = \langle \psi | f^n(P) X_n (P^2 - M^2) \psi \rangle - \langle X_n f^n(P)(P^2 - M^2) \psi | \psi \rangle = -2i \langle \psi | f^n(P) P_n \psi \rangle \]
Death of the Lightcone String

\[ [P^+, X^-] = i \, , \, M^2 = P^+ P^- - \vec{P}_T^2 \, , \, P^- = \frac{m^2 + \vec{P}_T^2}{P_+} \]

\( P^- \) not smooth in the spectrum of \( P^+ \)

\[ V_b = e^{iX^- b} \, , \, (V_b \Psi)(p_T, p^+) = \Psi(p_T, p^+ + b) \, , \]

\[ \Psi \neq 0 \rightarrow |(m^2 + p_T^2)\Psi(p_T, -b)| > \epsilon \text{ in some } \mathcal{U}_{-b} \]

\[ \langle P^-V_b\Psi|P^-V_b\Psi\rangle > \int_{\mathcal{U}} dp^+ d^{D-2} p \frac{\epsilon^2}{(p^+)^2} = \infty \]

There is no common, invariant domain of \( P^- \) and \( V_b \). They do not generate an algebra. The calculation of \( D = 26 \) is meaningless.

The algebra of the lightcone string has no domain.

Though the points with \( P^+ = 0 \) have measure 0, the singularity matters – hard to see in terms of \(|p_1, s_1, \ldots p_n, s_n\rangle\).
Contradiction

If the lightcone string represented rotations unitarily (generators $M_{ij}$), then $\vec{X} = (X_1, \ldots X_{D-1})$ existed together with $\vec{P} = (P^1, \ldots P^{D-1})$.

They are Heisenberg pairs and vectors under $M_{ij}$ and $L_{ij} = X_i P_j - X_j P_i$.

So $S_{ij} = M_{ij} - L_{ij}$ commutes with $X_i$ and $P_j$ and therefore with $L_{ij}$

$L_{ij}$ and $M_{ij} = L_{ij} + S_{ij}$ satisfy the angular momentum algebra of $SO(D - 1)$

$\Rightarrow S_{ij}$ satisfies the angular momentum algebra of $SO(D - 1)$.

? Massless states of the lightcone string are only helicity multiplets of $SO(D - 2)$!
**Massless Representation**  \( p^0 = |\vec{p}| > 0 \)

\( \mathcal{M}_0 = S^2 \times \mathbb{R} \neq \mathbb{R}^3 \), stabilizer \( E(2) \neq SO(3) \), no exercise for students

States: sections rather than functions, coordinate patches, not defined at \( p = 0 \),

Wigner rotation calculated in \( SL(2, \mathbb{C}) \), \( \hbar \) helicity

\[
\begin{align*}
(-iM_{12}\Psi)_{N}(p) &= -\left(p_x \partial_{p_y} - p_y \partial_{p_x}\right)\psi_N(p) - i\hbar \psi_N(p), \\
(-iM_{31}\Psi)_{N}(p) &= -\left(p_z \partial_{p_x} - p_x \partial_{p_z}\right)\psi_N(p) - i\hbar \frac{p_y}{|\vec{p}| + p_z} \psi_N(p), \\
(-iM_{32}\Psi)_{N}(p) &= -\left(p_z \partial_{p_y} - p_y \partial_{p_z}\right)\psi_N(p) + i\hbar \frac{p_x}{|\vec{p}| + p_z} \psi_N(p), \\
(-iM_{01}\Psi)_{N}(p) &= |\vec{p}| \partial_{p_x} \psi_N(p) - i\hbar \frac{p_y}{|\vec{p}| + p_z} \psi_N(p), \\
(-iM_{02}\Psi)_{N}(p) &= |\vec{p}| \partial_{p_y} \psi_N(p) + i\hbar \frac{p_x}{|\vec{p}| + p_z} \psi_N(p), \\
(-iM_{03}\Psi)_{N}(p) &= |\vec{p}| \partial_{p_z} \psi_N(p)
\end{align*}
\]
Singularity at \( A_\vDash \{ | \vec{p}|(1, 0, 0, -1) : |\vec{p}| > 0 \} \)

Outside \( A_\vDash \) the generators \( M_{kl} \) satisfy the Lorentz algebra for each value of \( \hbar \)

Smooth \( \Psi_N \) with \( \Psi_N|_{A_\vDash} \neq 0 \) not in the domain, e.g. \( \langle M_{13} \Psi_N|M_{13} \Psi_N \rangle = \infty \)

\[
\Psi_S(p) = e^{2i \hbar \varphi(p)} \Psi_N(p), \quad e^{2i \hbar \varphi(p)} = \left( (p_x + ip_y)/\sqrt{p_x^2 + p_y^2} \right)^{2h}, \quad 2h \in \mathbb{Z}
\]

\[
(-iM_{12})_S(p) = -(p_x \partial_{p_y} - p_y \partial_{p_x})\Psi_S(p) + i \hbar \Psi_S(p),
\]

\[
(-iM_{31})_S(p) = -(p_z \partial_{p_x} - p_x \partial_{p_z})\Psi_S(p) - i \hbar \frac{p_y}{|\vec{p}| - p_z} \Psi_S(p),
\]

\[
(-iM_{32})_S(p) = -(p_z \partial_{p_y} - p_y \partial_{p_z})\Psi_S(p) + i \hbar \frac{p_x}{|\vec{p}| - p_z} \Psi_S(p),
\]

\[
(-iM_{01})_S(p) = |\vec{p}| \partial_{p_x} \Psi_S(p) + i \hbar \frac{p_y}{|\vec{p}| - p_z} \Psi_S(p),
\]

\[
(-iM_{02})_S(p) = |\vec{p}| \partial_{p_y} \Psi_S(p) - i \hbar \frac{p_x}{|\vec{p}| - p_z} \Psi_S(p),
\]

\[
(-iM_{03})_S(p) = |\vec{p}| \partial_{p_z} \Psi_S(p)
\]
Noncommutative Geometry

\[ D_i = \partial p_i + A_i - \frac{p_i}{2|\vec{p}|^2} \]

\[ \vec{A}_N(p) = \frac{i \hbar}{|\vec{p}|(|\vec{p}| + p_z)} \begin{pmatrix} -p_y \\ p_x \\ 0 \end{pmatrix}, \quad \vec{A}_S(p) = \frac{i \hbar}{|\vec{p}|(|\vec{p}| - p_z)} \begin{pmatrix} p_y \\ -p_x \\ 0 \end{pmatrix} \]

\[ [P^i, P^j] = 0, \quad [P^i, D_j] = -\delta^i_j, \quad [D_i, D_j] = F_{ij} = \partial_p A_j - \partial_p A_i = i \hbar \epsilon_{ijk} \frac{p^k}{|\vec{p}|^3} \]

The geometry of massless particles with nonvanishing helicity is noncommutative.

\[ -iM_{ij} = -(P^iD_j - P^jD_i) - i \hbar \epsilon_{ijk} \frac{p^k}{|\vec{p}|}, \quad -iM_{0i} = |\vec{p}|^{1/2}D_i|\vec{p}|^{1/2} \]

Momentum Space Monopole: \[ \frac{1}{4\pi} \int_S \mathbf{F} = i \hbar \]
Scattering

Product rule ⇔ free time evolution: \((P^m \Psi)^{ij}(p_1, p_2) = (p_1^m + p_2^m) \Psi^{ij}(p_1, p_2)\)

Nontrivial group of unitary motion has no limit!
\[\Psi(s + t) = U'(s)\Psi(t), \Psi_\pm = U'(s)\Psi_\pm, \Psi(t) - \Psi_\pm = U'(t)(\Psi(0) - \Psi_\pm)\]

\[\Omega_\pm = \text{s-lim}_{t \to \mp \infty} e^{iHt} e^{-iH't}, S = \Omega_+ \Omega_-^{-1} = \text{s-lim}_{t, t' \to \infty} e^{-iHt} e^{iH'(t+t')} e^{-iH't'}, [H, H'] \neq 0\]

\[H\Omega_+ = \Omega_+ H' \text{ not reconstructible from Wightman distributions}\]
Schrödinger Picture

Interaction concerns time evolution of many particle states.

Language and pictures matter, e.g. Ptolemaic system.

Not the states but their time evolution is interacting. In, out and interacting are relations not properties. $\Psi_{\text{out}}$ is the initial state of a future asymptote, not a state with all momenta outwards as opposed to $\Psi_{\text{in}}$ with all momenta inwards.

States are just states, they determine probabilities of results.

Interacting fields, whatever they are, are not needed.

Lorentz transformations transform states and consequently paths irrespective of their time evolution: $M'_{kl} = M_{kl}$.

Claim: $P'^{m} = u^{m}M'$, $P^{m} = u^{m}M$, $M = \sqrt{P^{2}}$, $[u^{m}, M'] = 0$, $[M', M] \neq 0$.

? Canonical Quantization, Haag’s Theorem
Center Variables

\((p_1, \ldots p_n) \leftrightarrow (u, q_1, \ldots q_n) : \sum_i q_i = 0, q_i \in \mathbb{R}^3\)

\[ u = \frac{\sum_i p_i}{m}, \quad m^2 = (\sum_i p_i)^2, \quad u^2 = 1, \]

\[ p_i\| = (p_i \cdot u) u, \quad p_i\perp = p_i - p_i\|, \quad q_i = (L_u)^{-1}p_i\perp \]

\[ p_i(u, q) = \sqrt{m_i^2 + \bar{q}_i^2} u + L_u q_i = \sqrt{m_i^2 + \bar{q}_i^2} \left( \sqrt{1 + \bar{u}^2} \right) + \left( \bar{q}_i + \frac{\bar{u} \cdot \bar{q}_i}{1 + \sqrt{1 + \bar{u}^2}} \right) \]

\[ M = \sum_i \sqrt{m_i^2 + (\bar{q}_i)^2} \]

\[ H_2 = \sqrt{1 + \bar{u}^2} \left( \sqrt{m_1^2 + \bar{q}^2} + \sqrt{m_2^2 + \bar{q}^2} \right) = m_1 + m_2 + \frac{(m_1 + m_2)\bar{u}^2}{2} + \frac{\bar{q}^2}{2\mu} + \ldots \]
Generalized Wave Operators

\[
\lim_{t \to \infty} \left( e^{i \sqrt{1 + \vec{u}^2} M t} e^{-i H' t} \Psi \right) (u, q) = \lim_{t' \to \infty} \left( e^{i M t' e^{-i H' t' / \sqrt{1 + \vec{u}^2}}} \Psi \right) (u, q)
\]

H' / \sqrt{1 + \vec{u}^2} hermitian, if [H', u^0] = 0 \Rightarrow H' = u^0 M' \Rightarrow P'^m = u^m M'

\[
\Omega_{\pm}(H, H') = s\text{-lim}_{t \to \pm \infty} e^{i \sqrt{1 + \vec{u}^2} M t} e^{-i \sqrt{1 + \vec{u}^2} M' t} = s\text{-lim}_{t \to \mp \infty} e^{i M t} e^{-i M' t} = \Omega_{\pm}(M, M')
\]

M' = M + V, \ [V, M] \neq 0, V \text{ translational and rotational invariant } (\sum q_i = 0),

Invariance Principle:

\( f \) monotonously increasing \Rightarrow \( \Omega_{\pm}(M, M') = \Omega_{\pm}(f(M), f(M')) \)

Relativistic and Nonrelativistic scattering and binding analogous
Particles not localized in the forward and backward cone of a compact domain.

Boosts transform states \((\neq \text{fields})\) nonlocally by convolution.

The state \(\int_{\mathcal{U}} d^4x \, f(x) \, \Phi(x) \, \Omega\), \(\Phi(x)\) a local field, is not localized within \(\mathcal{U}\).

Hilbert space does not allow for a massless vector field. Time order of a massive vector field does not yield a covariant propagator

\[
\langle T \, \Phi^m(x) \, \Phi^n(0) \rangle = \lim_{\epsilon \to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{\eta^{mn} + p^m p^n / m^2}{p^2 - m^2 + i\epsilon} e^{ipx}
\]

The \(1/m^2\)-term is not cancelled by adding to \(\Phi^m\) a gradient \(\partial^m \phi\).
Massless particles have no position operator.

Massless fields have algebraically special Fourier transformation of rank 1

\[ \Phi_{\alpha_1...\alpha_{2h}}(x) = \int \tilde{d}p \ n_{\alpha_1}(p) \ldots n_{\alpha_{2h}}(p) \left( e^{ip \cdot x} b_N^*(p) + e^{-ip \cdot x} a_N(p) \right) \]

\[ = \int \tilde{d}p \ s_{\alpha_1}(p) \ldots s_{\alpha_{2h}}(p) \left( e^{ip \cdot x} b_S^*(p) + e^{-ip \cdot x} a_S(p) \right) \]

Antiparticles not because of CPT but if real representations \( \hat{D} \) of internal symmetries decompose over complex numbers, \( \hat{D} = D + D^* \).

Though Wigner’s fundamental paper celebrates its 80\(^{th}\) birthday basic questions are still unresolved, e.g. the relation of \( M' \) to \( \mathcal{L}_{\text{int}} \).