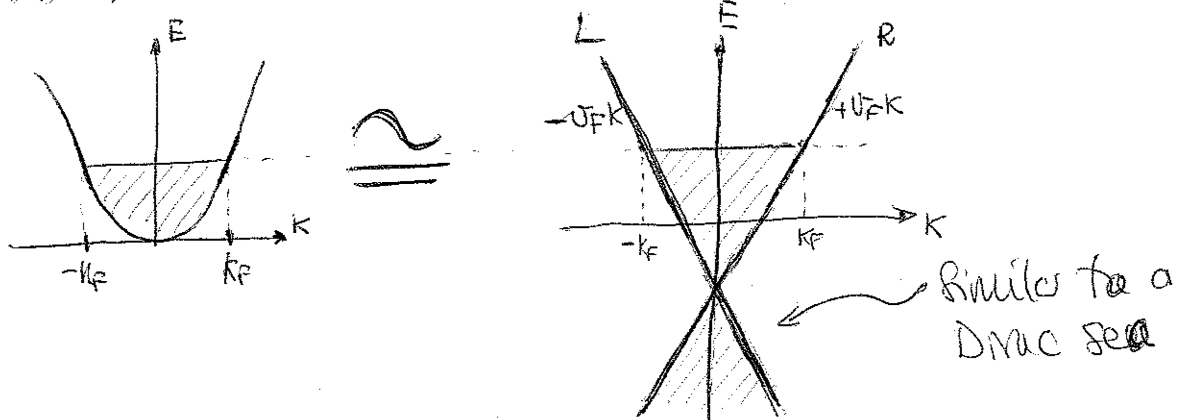
• THE BASICS OF BOSONIZATION I : SPIN-LESS FERMIONS

* We shall start our discussion with the (simpler) case of spin-less fermions.

* Tomonaga-Luttinger model

* We just saw that particle-hole excitations have a nearly linear spectrum, and are essentially well-defined excitations, with a well-defined energy and momentum.
* In order to make this relation perfect we replace the original model by one in which the spectrum is purely linear (Tomonaga-Luttinger model)
* To get a total independence of the energy of the particle-hole pair on the initial momentum for all q we must extend the energy spectrum to $-\infty$



* This forces us to introduce two species of fermions:
  * Right-going fermions: with dispersion $+v_F K$
  * Left-going fermions: with dispersion $-v_F K$

* The Hamiltonian of the system becomes:

$$H = \sum_{r=R,L} \sum_K v_F\,(rK - k_F)\; c^\dagger_{r,K}\, c_{r,K}$$

This r is $\begin{cases} +1 & \text{for } r=R \leftarrow \text{RIGHT-going} \\ -1 & \text{for } r=L \leftarrow \text{LEFT-going} \end{cases}$

(Note: This equation is a sort of 1D equivalent of a Dirac Hamiltonian)

* The particle-hole excitations are now (e.g. for right-movers):

$$E_{R,k}(q) = v_F(k+q) - v_F k = v_F q$$

i.e. independent of q.

The particle-hole excitations are hence well defined

* Particle-hole excitations

* Since particle-hole excitations (which give density fluctuations) are well-defined we will employ them for the description of the system.
* Density fluctuations are given by superpositions of particle-hole fluctuations:

$$\rho^\dagger(q) = \sum_k c^\dagger_{k+q} c_k$$

* Before entering into the detailed discussion of density fluctuations let's see why they are so important.

Clearly they are bosonic, and well defined. They are extremelly useful when treating interactions. A typical interaction Hamiltonian is of the form: ($\Omega \equiv$ Quantization "volume")

$$V = \frac{1}{2\Omega} \sum_{k,k',q} V(q)\, c^\dagger_{k+q} c^\dagger_{k'-q} c_{k'} c_k$$

$$= \frac{1}{2\Omega} \sum_q V(q) \underbrace{\left[\sum_k c^\dagger_{k+q} c_k\right]}_{\rho(q)} \underbrace{\left[\sum_{k'} c^\dagger_{k'-q} c_{k'}\right]}_{\rho(-q)}$$

Hence a quartic Hamiltonian in Fermi operators (and hence a difficult problem) becomes a quadratic Hamiltonian in the density fluctuations (and hence a much simpler problem to solve!).

* Normal ordering

- Due to the "Dirac sea" in p. 5 we have to be careful.
One introduces normal ordering, which for 2 operators A and B that are linear combinations of creation/annihilation operators amounts to:

$$:AB: \; = AB - \langle 0|AB|0\rangle \qquad (|0\rangle \equiv \text{vacuum})$$

- The normal ordered density is then (with $r = R, L$)

$$:\rho_r(x): \; = \; :\psi_r^+(x)\psi_r(x): \quad \longleftarrow \text{operator creating a right or left mover.}$$

$$\psi_r(x) \equiv \frac{1}{\sqrt{\Omega}} \sum_k e^{ikx} c_{rk}$$

- Fourier-transforming:

$$:\rho_r(x): \; = \frac{1}{\Omega} \sum_p :\rho_r(p): e^{ipx}$$

where:

$$:\rho_r(p): \; = \begin{cases} \sum_k c^+_{r,k+p} c_{r,k} & (p \neq 0) \\ = N_r \equiv \sum_k \left[c^+_{rk} c_{rk} - \langle 0| c^+_{r,k} c_{r,k} |0\rangle \right] & (p = 0) \end{cases}$$

These operators appear later quite often $\longrightarrow$

(Note: $|0\rangle$ is the ground state of the Tomonaga-Luttinger Hamiltonian p. 5)

Note also that $\rho^+(p) = \rho(-p)$ (since $\rho(x)$ is real)

- Note that the substraction of the average in vacuum ensures that the density operator remains finite, despite the infinite occupation of the "Dirac sea".

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* Commutator of the density operators: Bosonic operators

- Obviously $[\rho_R^+(p), \rho_L^+(p')] = 0$
- For identical species

$$[\rho_r^+(p), \rho_r^+(-p')] = \sum_{k_1, k_2} \left[c^+_{r,k_1+p} c_{r,k_1} \, , \, c^+_{r,k_2-p'} c_{r,k_2} \right]$$

$$= \sum_{k_1, k_2} \left[c^+_{r k_1, p} c_{r, k_2} \delta_{k_1, k_2 - p'} - c^+_{r, k_2 - p'} c_{r, k_1} \delta_{k_1 + p, k_2} \right]$$

$$= \sum_{k_2} \left[c^+_{r, k_2 + p - p'} c_{r, k_2} - c^+_{r, k_2 - p'} c_{r, k_2 - p} \right]$$

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- Naively, it would look as if the commutator is zero (it seems that one may change variable e.g. in the 2^{nd} term $k_2 - p \to k_2$, and then one gets zero).
- However one must remember our previous discussion on normal ordering, and the avoidance of infinities. One can only make a change of variable in <u>normal ordering</u>:

$$[\rho_r^+(p), \rho_r^+(-p')] = \sum_{k_2} \left(:c^+_{r,k_2+p-p'} c_{r,k_2}: - :c^+_{r,k_2-p'} c_{r,k_2-p}: \right)$$
$$+ \sum_{k_2} \left[\langle 0| c^+_{r,k_2+p-p'} c_{r,k_2}|0\rangle - \langle 0| c^+_{r,k_2-p'} c_{r,k_2-p}|0\rangle \right]$$

In the normal-ordered term we can safely change variable and get zero. And hence we get

$$[\rho_r^+(p), \rho_r^+(-p')] = \delta_{p,p'} \sum_{k_2} \left[\langle 0| c^+_{r,k_2} c_{r,k_2}|0\rangle - \langle 0| c^+_{r,k_2-p} c_{r,k_2-p}|0\rangle \right]$$

- For periodic boundary conditions: $k = \frac{2\pi}{L} n \leftarrow$ momentum quantized

in addition $\langle 0| c^+_{r,k_2} c_{r,k_2}|0\rangle = \begin{cases} 1 \to \text{state occupied} \\ 0 \to \text{not occupied} \end{cases}$

Then $\sum_{k_2} \langle 0| c^+_{r,k_2} c_{r,k_2}|0\rangle - \sum_{k_2} \langle 0| c^+_{r,k_2-p} c_{r,k_2-p}|0\rangle$

$$= \begin{cases} \sum_{k_2=-\infty}^{k_F} \langle 0| c^+_{R,k_2} c_{R,k_2}|0\rangle - \sum_{k_2=-\infty}^{k_F+p} \langle 0| c^+_{R,k_2-p} c_{R,k_2-p}|0\rangle = \dfrac{-pL}{2\pi} \\ \sum_{k_2=-k_F}^{+\infty} \langle 0| c^+_{L,k_2} c_{L,k_2}|0\rangle - \sum_{k_2=-k_F+p}^{\infty} \langle 0| c^+_{L,k_2-p} c_{L,k_2-p}|0\rangle = \dfrac{pL}{2\pi} \end{cases}$$

Hence $[\rho_r^+(p), \rho_r^+(-p')] = -\delta_{pp'}\, r \dfrac{pL}{2\pi}$ (recall that $r = \begin{cases} +1 \to R \\ -1 \to L \end{cases}$)

Summarizing: $\boxed{[\rho_r^+(p), \rho_{r'}^+(-p')] = -\delta_{r,r'} \delta_{pp'} \frac{rpL}{2\pi}}$

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- This result is crucial. Because of the infinite number of occupied states, the density operators behave (up to normalization) like bosonic operators (as we already pointed out intuitively before).

* Since $\rho_L^+(p>0)\,|0\rangle = 0$, $\rho_R^+(p<0)\,|0\rangle = 0$ } the density operators may be identified with the ladder operators of bosons:

$$\boxed{\left.\begin{cases} b_p^+ \equiv \left(\dfrac{2\pi}{L|p|}\right)^{1/2} \displaystyle\sum_r Y(rp)\,\rho_r^+(p) \\ b_p \equiv \left(\dfrac{2\pi}{L|p|}\right)^{1/2} \displaystyle\sum_r Y(rp)\,\rho_r^+(-p) \end{cases}\right\} \equiv \text{BOSONIC OPERATORS (only defined for } p\neq 0)}$$

where $Y(x)$ is the Heaviside function ($Y(x)=0$ $x<0$, $Y(x)=1$ $x>0$)

* Hamiltonian in terms of the boson operators
* Let's have a look to the Hamiltonian itself. We want the commutator $[b_{p_0}, H]$ with say $p_0>0$:

$$[b_{p_0}, H] = \left(\frac{2\pi}{L|p_0|}\right)^{1/2} \sum_{r,k} \left[\rho_R^+(-p_0), v_F(rk-k_F)\, c_{rk}^+ c_{rk}\right]$$

$$= \left(\frac{2\pi}{L|p_0|}\right)^{1/2} \sum_{k,k_1} v_F(k-k_F) \left[c_{R,k_1-p_0}^+ c_{R,k_1},\, c_{Rk}^+ c_{Rk}\right]$$

$$= \left(\frac{2\pi}{L|p_0|}\right)^{1/2} \sum_{k,k_1} v_F(k-k_F) \Big[\underbrace{c_{R,k_1-p_0}^+ c_{Rk_1} c_{Rk}^+ c_{Rk}}_{\Downarrow} - c_{Rk}^+ c_{Rk}\, c_{R,k_1-p_0}^+ c_{R,k_1}\Big]$$

$$c_{Rk}^+ c_{Rk}\, c_{Rk_1-p_0}^+ c_{R,k_1} + c_{R,k_1-p_0}^+ c_{Rk}\,\delta_{kk_1} - c_{Rk}^+ c_{Rk_1}\,\delta_{k,k_1-p_0}$$

$$= \left(\frac{2\pi}{L|p_0|}\right) \sum_k v_F(k-k_F)\left[c_{R,k-p_0}^+ c_{R,k} - c_{Rk}^+ c_{R,k+p_0}\right]$$

$$= \left(\frac{2\pi}{L|p_0|}\right) \sum_k v_F\, p_0\, c_{R,k-p_0}^+ c_{R,k} = v_F p_0 \left[\left(\frac{2\pi}{L|p_0|}\right)^{1/2} \sum_k c_{R,k-p_0}^+ c_{R,k}\right]$$

Hence $\boxed{[b_{p_0}, H] = v_F p_0\, b_{p_0}}$

and similarly for $p<0$ and b_p^+.

Hence $$\boxed{H \simeq \sum_{p\neq 0} v_F |p| \, b_p^+ b_p}$$

* Hence the kinetic energy (which is what is given by the Luttinger Hamiltonian) is quadratic in the boson operators. (Note: this is somehow unexpected and remarkable, since it is also quadratic in the fermion operators!)
* Since the interaction energy is also quadratic (recall p. 6), then the whole Hamiltonian will remain quadratic, and hence remarkably simple!

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* Single particle creation operators

· Recall (p. 7) $\psi_r(x) = \frac{1}{\sqrt{\Omega}} \sum_k e^{ikx} c_{r,k}$

Then $[\rho_r^+(p), \psi_r(x)] = \frac{1}{\sqrt{\Omega}} \sum_{k,k_1} e^{ik_1 x} [c^+_{r,k+p} c_{r,k}, c_{r,k_1}]$

$= -\frac{1}{\sqrt{\Omega}} \sum_{k,k_1} e^{ik_1 x} c_{r,k}\, \delta_{k_1,k+p} = -e^{ipx} \left[\frac{1}{\sqrt{\Omega}} \sum_k e^{ikx} c_{r,k}\right] = -e^{ipx}\psi_r(x)$

* An operator written in terms of boson operators and that would produce the same commutation is
$$\psi_r(x) \cong e^{\sum_p e^{ipx} \rho_r^+(-p) \left(\frac{2\pi r}{pL}\right)}$$

$[A, f(B)] = [A,B] f'(B)$

let's check it:

$\left[\rho_r^+(p), \exp\left[\sum_{p'} e^{ip'x} \rho_r^+(-p')\left(\frac{2\pi r}{p'L}\right)\right]\right] =$

$= \sum_{p'} e^{ip'x} \left(\frac{2\pi r}{p'L}\right) [\rho_r^+(p), \rho_r^+(-p')] \exp\left[\sum_{p'} e^{ip'x} \rho_r^+(-p')\left(\frac{2\pi r}{p'L}\right)\right]$

$= -e^{ipx} \left[\sum_{p'} e^{ip'x} \rho_r^+(-p')\left(\frac{2\pi r}{p'L}\right)\right]$ as we wanted.

* Any fermionic operator can now be written in the boson language!

* Klein factors

* We have expressed $\hat{A}$ and $\psi_r(x)$ as a function of the boson operators. However those expressions we found cannot be fully correct. The reason is simple:
  - $\psi_r(x)$ changes the total number of r-fermions by one
  - b and $b^\dagger$ preserve the number of fermions of each r (they are only density fluctuations)

* To solve this problem we must add two additional operators U_r (so called Klein factors) that change the total number of fermions:
  - $U_r^\dagger$ adds one fermion r
  - U_r commutes with the boson operators

* The Klein operators are of the form:

$$U_r^\dagger \equiv \frac{1}{\sqrt{L}} \int_0^L dx\, e^{irk_F x}\, e^{-i\phi_r^\dagger(x)}\, \psi_r^\dagger(x)\, e^{-i\phi_r(x)}$$

where we employ the so-called chiral fields

$$\phi_r(x) \equiv -\frac{\pi r x}{L} N_r + \lim_{\epsilon\to 0} i \sum_{p\neq 0} \left(\frac{2\pi}{L|p|}\right)^{1/2} e^{-L\epsilon|p|/2\pi}\, Y(rp)\, b_p\, e^{ipx}$$

(Note: we won't proof that $U_r^\dagger$ works indeed in the way desired, for that have a look to Giamarchi's book, Appendix B.1.)

* We will provide in a moment an expression for $\hat{A}$ and ψ_r that contains the necessary corrections. We will see later on, however, that these corrections are most of the times ~~negli~~ not too relevant physically.

* Note: The Klein factors fullfill ~~that~~ U_r of different species anticommute, whereas for the same species $UU^\dagger = U^\dagger U = 1$.

* The ϕ and θ fields

• We will introduce at this point two operators, ϕ and θ, that will play a key role in all our future discussion:

$$\phi(x) = -(N_R+N_L)\frac{\pi x}{L} - \frac{i\pi}{L}\sum_{p\neq 0}\frac{1}{p}e^{-\alpha|p|/2-ipx}\left(\rho_R^\dagger(p)+\rho_L^\dagger(p)\right)$$

$$\theta(x) = (N_R-N_L)\frac{\pi x}{L} + \frac{i\pi}{L}\sum_{p\neq 0}\frac{1}{p}e^{-\alpha|p|/2-ipx}\left(\rho_R^\dagger(p)-\rho_L^\dagger(p)\right)$$

• Using these fields or the boson operators we can finally write the exact form of $\hat{H}$ and ψ_r:

$$\hat{H} = \sum_{p\neq 0} v_F|p|\, b_p^\dagger b_p + \frac{\pi v_F}{L}\sum_r N_r^2$$

$$\psi_r(x) = U_r \lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi\alpha}}\, e^{ir(k_F-\frac{\pi}{L})x}\, e^{-i(r\phi(x)-\theta(x))}$$

where α is a cut-off

(Notes: • The terms with N_r can be understood as the $p\to 0$ limit of the boson terms
• α is an arbitrary cut-off which prevents the momentum to become too large (factor $e^{-\alpha|p|/2}$ above)

* Since $b_p^\dagger \equiv \left(\frac{2\pi}{L|p|}\right)^{1/2}\sum_r Y(rp)\rho_r^\dagger(p)$ and $b_p \equiv \left(\frac{2\pi}{L|p|}\right)^{1/2}\sum_r Y(rp)\rho_r^\dagger(-p)$

then:

$$\boxed{\begin{aligned}\phi(x) &= -(N_R+N_L)\frac{\pi x}{L} - \frac{i\pi}{L}\sum_{p\neq 0}\left(\frac{L|p|}{2\pi}\right)^{1/2}\frac{1}{p}e^{-\alpha|p|/2-ipx}\left(b_p^\dagger+b_{-p}\right)\\ \theta(x) &= (N_R-N_L)\frac{\pi x}{L} + \frac{i\pi}{L}\sum_{p\neq 0}\left(\frac{L|p|}{2\pi}\right)^{1/2}\frac{1}{|p|}e^{-\alpha|p|/2-ipx}\left(b_p^\dagger-b_{-p}\right)\end{aligned}}$$

(we keep α finite to introduce a momentum cut-off)

• The fields $\phi(x)$ and $\theta(x)$ have simple physical interpretations that we will discuss later in these lectures.

* let's have a look to the commutation relations of ϕ and Θ:

$$[\phi(x_1),\Theta(x_2)] = \frac{\pi^2}{L^2}\sum_{p,p'}\frac{L}{2\pi}\sqrt{|p||p'|}\,\frac{1}{p|p'|}\,e^{-\frac{\alpha}{2}(|p|+|p'|)-i(px_1+p'x_2)}\underbrace{[b_p^+ + b_{-p}, b_{p'}^+ - b_{-p'}]}_{2\delta_{p',-p}}$$

$$= \frac{\pi}{L}\sum_p \frac{1}{p}e^{-\alpha|p|}e^{-ip(x_1-x_2)} \xrightarrow{L\to\infty}$$

$$\longrightarrow \int_{-\infty}^{\infty} dp\, \frac{e^{-\alpha|p|}}{p}\, e^{-ip(x_1-x_2)} = i\int_0^\infty \frac{dp}{p}\sin[p(x_2-x_1)]\, e^{-\alpha|p|}$$

$$\xrightarrow{\alpha\to 0} i\frac{\pi}{2}\operatorname{sign}(x_2-x_1)$$

$$\Longrightarrow \boxed{[\phi(x_1),\Theta(x_2)] = i\frac{\pi}{2}\operatorname{sign}(x_2-x_1)}$$

* Similarly, here $\nabla\equiv\partial_x$

$$[\phi(x_1),\nabla\Theta(x_2)] = i\int_0^\infty dp\,\cos[p(x_2-x_1)]\,e^{-\alpha|p|} \overset{\alpha\to 0}{=} i\pi\,\delta(x_2-x_1)$$

Hence $\boxed{\Pi(x)\equiv\frac{1}{\pi}\nabla\Theta(x)}$ is the conjugate momentum to the field $\phi(x)$

* Other quite interesting point that provides physical insight into the meaning of Θ and ϕ is that (for $L\to\infty$)

$$\boxed{\begin{aligned}\nabla\phi(x) &= -\pi[\rho_R(x)+\rho_L(x)]\\ \nabla\Theta(x) &= \pi[\rho_R(x)-\rho_L(x)]\end{aligned}}\quad\Bigg\}\ \text{(This is trivial to obtain from the expressions in p. 12)}$$

* Hence $\nabla\phi$ is related to the sum of left and right density fluctuations. Hence it's related to the $q\sim 0$ density fluctuations.
* On the contrary $\nabla\Theta$ counts the difference between left and right movers $\longrightarrow$ Π is hence the current operator

• Finally we can write the Hamiltonian in terms of θ and ϕ. (14)
Removing (once more) terms vanishing when $L \to \infty$ (thermodynamic limit)

$$\boxed{H = \frac{1}{2\pi} \int dx \, v_F \left\{ (\pi \Pi(x))^2 + (\nabla \phi(x))^2 \right\}}$$

(Note: when we take the $L \to \infty$ limit the naive form $\sum v_F |p| b_p^\dagger b_p$ is recovered, and from the definition of b_p and $b_p^\dagger$ and Π and $\nabla\phi$ we recover the previous form of the Hamiltonian)

* Similarly

$$\boxed{\psi_r(x) = \underbrace{e^{irk_F x}}_{\text{rapid oscillation}} \; \underbrace{\tilde{\psi}_r(x)}_{\text{varies slowly at the scale of } k_F^{-1}} = \frac{U_r}{\sqrt{2\pi\alpha}} e^{irk_F x} e^{-i(r\phi(x) - \theta(x))}}$$

This separation of oscillation scales will be very useful in our future discussions.

• Interaction Hamiltonian

• Up to now we have just considered in detail the kinetic energy (the Tomonaga-Luttinger Hamiltonian of p. 5). We will now consider the interparticle interactions.

• For spin-less fermions a typical interaction is of the form:

$$H = \int dx \int dx' \, \rho(x) \, V(x-x') \, \rho(x')$$

[for the moment we shall consider only contact interactions $V(x) \sim \delta(x)$]

where $\rho(x)$ is the density, but since we have now right and left movers we have first to consider what is actually $\rho(x)$.

$$\rho(x) = \psi^\dagger(x) \psi(x)$$

where $\psi(x) = \frac{1}{\sqrt{\Omega}} \sum_k e^{ikx} c_k$

Since only the part of the single-particle operator acting close to the Fermi surface is important for the low-energy properties, we may write then:

$$\psi(x) \simeq \frac{1}{\sqrt{\Omega}}\Big[\underbrace{\sum_{-\Lambda<k-k_F<\Lambda} e^{ikx} c_k}_{\text{Right movers}} + \underbrace{\sum_{-\Lambda<k+k_F<\Lambda} e^{ikx} c_k}_{\text{Left movers}}\Big]$$

where Λ is a momentum cut-off

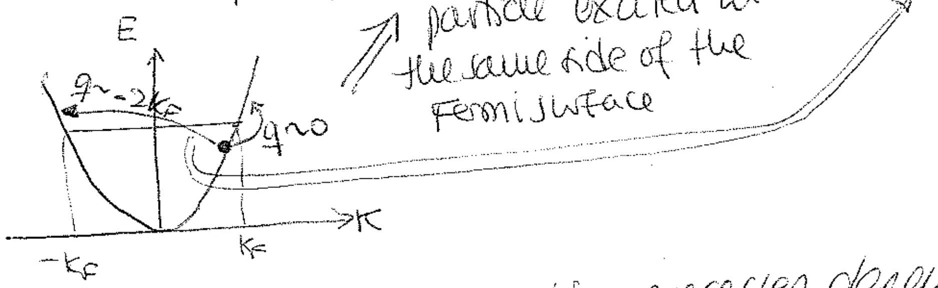
Hence $\psi(x) = \psi_R(x) + \psi_L(x)$

and the density becomes:

$$\rho(x) = \underbrace{\psi_R^\dagger(x)\psi_R(x) + \psi_L^\dagger(x)\psi_L(x)}_{q\sim 0 \text{ part of the density}} + \underbrace{\psi_R^\dagger(x)\psi_L(x) + \psi_L^\dagger(x)\psi_R(x)}_{q\sim \pm 2k_F \text{ part}}$$

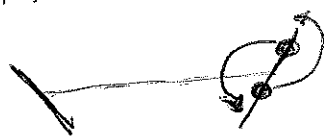
particle excited at the same side of the Fermi surface

particle excited at the opposite side of the Fermi surface.



* One has different interaction processes depending on whether the incoming or outgoing fermions are right or left movers:

* g_4 processes : only couple fermions on the same side of the Fermi surface $\Rightarrow$

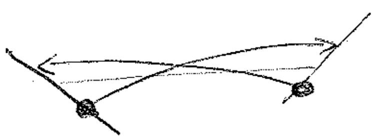


(characterized by a coupling constant g_4)

* g_2 processes : couple fermions from one side with fermions on the other side. However each species stays on the same side of the Fermi surface after the interaction:

(characterized by a coupling constant g_2)

* g_1 processes : corresponds to a backscattering ($\pm 2k_F$) where fermions exchange sides



Note that for spinless fermions g_1 and g_2 processes are identical since the particles are indiscernible. This is not the case for spinfull fermions where g_2 and g_1 are different.

* Let's have a look first to the g_4 processes:

$$\nabla\phi = -\pi(\rho_R+\rho_L)$$
$$\nabla\theta = \pi(\rho_R-\rho_L)$$
$$\Downarrow$$
$$\rho_R = \frac{1}{2\pi}(\nabla\theta-\nabla\phi)$$

$$\frac{g_4}{2}\psi_R^+(x)\psi_R(x)\psi_R^+(x)\psi_R(x) = \frac{g_4}{2}\rho_R(x)\rho_R(x) =$$

$$= \frac{g_4}{2}\frac{1}{(2\pi)^2}(\nabla\phi-\nabla\theta)^2$$

For left movers:

$$\frac{g_4}{2}\psi_L^+(x)\psi_L(x)\psi_L^+(x)\psi_L(x) = \frac{g_4}{2}\rho_L(x)\rho_L(x) = \frac{g_4}{2}\frac{1}{(2\pi)^2}(\nabla\phi+\nabla\theta)^2$$

Adding both terms we arrive at the g_4 part of the interaction Hamiltonian:
$$\frac{g_4}{(2\pi)^2}\int dx\,\left[(\nabla\phi)^2+(\nabla\theta)^2\right]$$

* Recall (p. 14) that the kinetic energy may be written as well in the same form $\frac{v_F}{2\pi}\int dx\,[(\nabla\phi)^2+(\nabla\theta)^2]$. Hence the g_4 processes just renormalize the velocity of the excitations, which becomes

$$v_F \longrightarrow u = v_F\left[1+g_4/\pi v_F\right]$$

* Let's consider now the g_2 processes

$$g_2\psi_R^+(x)\psi_R(x)\psi_L^+(x)\psi_L(x) = g_2\rho_R(x)\rho_L(x)$$

$$= \frac{g_2}{(2\pi)^2}(\nabla\theta-\nabla\phi)(-\nabla\theta-\nabla\phi) = \frac{g_2}{(2\pi)^2}\left[(\nabla\phi)^2-(\nabla\theta)^2\right]$$

* Contrary to the g_4 processes the g_2 processes change the relative weights of $(\nabla\phi)^2$ and $(\nabla\theta)^2$ in the Hamiltonian, but the crucial point is that the Hamiltonian remains quadratic even in the presence of interactions.

* Due to the asymmetry introduced by the g_2 processes we shall need now <u>two</u> parameters to describe the system (and not only the velocity of excitations).

* The quadratic Hamiltonian may be re-written in the form:

$$\boxed{H = \frac{1}{2\pi}\int dx \left\{ uK\,(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\phi(x))^2 \right\}}$$

where $\boxed{u = v_F\left[\left(1+\frac{y_4}{2}\right)^2 - \left(y_2/2\right)^2\right]^{1/2}}$ is the velocity of excitations

and $\boxed{K = \left(\frac{1+y_4/2-y_2/2}{1+y_4/2+y_2/2}\right)^{1/2}}$ is the Luttinger-parameter

where $y_2 \equiv g_2/\pi v_F$, $y_4 \equiv g_4/\pi v_F$

. Note that $\begin{cases} K<1 & \text{for } y_2>0 \text{ (repulsive interactions)} \\ K>1 & \text{for } y_2<0 \text{ (attractive interactions)} \end{cases}$

. So interacting spinless fermions can be described by free bosonic excitations.
This is a huge simplification indeed!

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* Thermodynamics

. Since the interacting system is still described by free bosons (Note: from the expressions of p. (12) we see that we can re-define $\tilde{\phi} \equiv \phi K$ and $\tilde{\theta} \equiv \theta/K$, and re-define in this way b_p and $b_p^\dagger$, and we will still get a Hamiltonian like in p. (10) but with u instead of v_F)

we can still write the spectrum in the form: (where we use periodic boundary conditions)

$$\epsilon(p) = u|p| = u\,\frac{2\pi}{L}|n|$$

* We may then calculate for example the specific heat:

$$C_v = \frac{dE}{dT} = \frac{d}{dT}\sum_{p\neq 0}\epsilon(p)\,f_B(\epsilon(p))$$

Bose distribution function $f_B(\epsilon) = \frac{1}{e^{\beta\epsilon}-1}$, $\beta = 1/T$

$$= -\beta^2 \sum_{p\neq 0}\epsilon(p)\frac{d}{d\beta}\left[\left(e^{\beta\epsilon(p)}-1\right)^{-1}\right]$$

$$= -\beta^2\sum_{p\neq 0}\epsilon(p)\left[(-1)\,\epsilon(p)\left(e^{\beta\epsilon(p)}\right)\left(e^{\beta\epsilon(p)}-1\right)^{-2}\right]$$

$$= \beta^2\sum_{p\neq 0}\epsilon(p)^2\frac{e^{\beta\epsilon(p)}}{\left(e^{\beta\epsilon(p)}-1\right)^2} = \frac{1}{T^2}\sum_{p\neq 0}u^2|p|^2\frac{e^{u|p|/T}}{\left[e^{u|p|/T}-1\right]^2} =$$

$$= \frac{u^2}{T^2} \sum_{p\neq 0} |p|^2 \frac{1}{\left[e^{u|p|/2T} - e^{-u|p|/2T}\right]^2} = \frac{u^2}{4T^2} \sum_{p\neq 0} \frac{p^2}{\sinh^2\left(\frac{u|p|}{2T}\right)} \Rightarrow$$

$$\xrightarrow{L\to\infty} \frac{u^2}{4T^2} \frac{L}{2\pi} \int dp \frac{p^2}{\sinh^2\left(\frac{up}{2T}\right)} = \frac{u^2}{4T^2} \frac{L}{2\pi} \left(\frac{8}{(\beta u)^3}\right) \underbrace{\int d\tilde{p} \frac{\tilde{p}^2}{\sinh^2 \tilde{p}}}_{\pi^2/3}$$

- Then:

$$C_V = \frac{u^2}{4T^2} \frac{L}{2\pi} \frac{\pi^2 8}{3u^3} T^3 = \frac{T}{u} \left(\frac{L\pi}{3}\right)$$

* For free fermions we have the same but with v_F instead of u.

$$C_V^0 = \frac{T}{v_F} \left(\frac{L\pi}{3}\right)$$

* So even in the presence of interaction $C_V \sim T$. The only difference is that $\boxed{C_V / C_V^0 = v_F / u}$

* Let's calculate now the compressibility
- We'll define it as $\kappa = \frac{\partial\rho}{\partial\mu}$ where μ is the chemical potential

(Note: strictly speaking κ should measure the relative change of volume (Ω) with pressure (P): $\kappa = -\frac{1}{\Omega}\frac{\partial\Omega}{\partial P} = \frac{1}{\rho^2}\frac{\partial\rho}{\partial\mu}$)

- Let's have a look now to the effect of a uniform chemical potential in our Hamiltonian. It adds a term:

$$-\mu \int dx\, \rho(x)$$

where $\rho(x) = \psi_R^\dagger(x)\psi_R(x) + \psi_L^\dagger(x)\psi_L(x) + \psi_R^\dagger(x)\psi_L(x) + \psi_L^\dagger(x)\psi_R(x)$

$$\xrightarrow{\text{p. (14)}} = \underbrace{\tilde{\psi}_R^\dagger(x)\tilde{\psi}_R(x) + \tilde{\psi}_L^\dagger(x)\tilde{\psi}_L(x)}_{\text{slowly varying}} + \underbrace{e^{-2ik_Fx}\tilde{\psi}_R^\dagger(x)\tilde{\psi}_L(x) + e^{2ik_Fx}\tilde{\psi}_L^\dagger(x)\tilde{\psi}_R(x)}_{\text{rapidly oscillating}}$$

slowly varying → They survive when integrating over x

rapidly oscillating → They dissappear when integrated over x

(Note: This procedure of keeping only slowly varying terms will be used a lot from now on!)

* Hence :

$$-\mu \int dx\, \rho(x) = -\mu \int dx\, (\rho_R(x) + \rho_L(x)) =$$ p. (13)

$$= \frac{\mu}{\pi} \int dx\, \nabla\phi(x)$$

* Let $\tilde{\phi} = \phi + \mu \frac{K}{u} x \longrightarrow \nabla\tilde{\phi} = \nabla\phi + \frac{\mu K}{u}$ This is the new term

$$\Rightarrow \frac{u}{2\pi K}(\nabla\tilde{\phi})^2 = \frac{u}{2\pi K}(\nabla\phi)^2 + \frac{\mu}{\pi}(\nabla\phi) + \frac{u}{2\pi K}\left(\frac{\mu K}{u}\right)^2$$

* Hence up to a constant we may rewrite the Hamiltonian in the form of p. (17) but with $\tilde{\phi} = \phi + \mu \frac{K}{u} x$

* We may calculate the compressibility from the average density

$$\frac{1}{L}\int dx\, \rho(x) = \frac{-1}{\pi L}\int dx\, \nabla\phi(x) = \underbrace{-\frac{1}{\pi L}\int dx\, \nabla\tilde{\phi}(x)}_{\langle \nabla\tilde{\phi} \rangle = 0} + \frac{1}{\pi L}\cdot\frac{\mu K}{u}\int dx$$

$\leftarrow$ they are the fluctuations in the renormalized Hamiltonian

Hence $\langle \rho(x) \rangle = \mu \frac{K}{\pi u}$

and hence $\kappa = \frac{d}{d\mu}\langle \rho(x) \rangle \Longrightarrow \boxed{\kappa = \frac{K}{\pi u}}$

. The parameter $\frac{K}{u}$ is hence directly linked to the compressibility (Note: This is a very useful relation to determine later on the Luttinger parameter K from the microscopic models).

* The compressibility of the free fermion gas is $\kappa_0 = \frac{1}{\pi v_F}$,

and hence $\boxed{\frac{\kappa}{\kappa_0} = v_F \frac{K}{u}}$

* Hence, the thermodynamic quantities look very much like for a non-interacting case : $C_V \sim T$ and κ is constant.

* Correlations

• Refresh on path integral formalism

- Before we start with the actual calculation of correlation functions for the quadratic Hamiltonian, let's have first a look to some key ideas of path integral formalism, which we will employ later.

• let's consider two canonically conjugated variables ϕ, π

$$[\phi(x), \pi(x')] = i\hbar\,\delta(x-x')$$

and a Hamiltonian $H(\phi,\pi)$. The partition function $Z = \mathrm{Tr}[e^{-\beta H}]$ may be expressed as a functional integral of the form:

$$Z = \int D\phi(x,\tau)\, D\pi(x,\tau) \exp\left\{\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int dx \left[i\pi(x,\tau)\partial_\tau\phi(x,\tau) - H(\phi(x,\tau),\pi(x,\tau))\right]\right\}$$

(Note: because of the trace and the bosonic character of the system $\phi(x,\tau+\beta) = \phi(x,\tau)$.)

• Let $\hat{O}$ and $\hat{A}$ be two operators. Time-ordered correlation functions can be also obtained from the functional integration

$$\langle T_\tau\, \hat{O}(\tau)\hat{A}(0)\rangle = \frac{1}{Z}\int D\phi(x,\tau)\, D\pi(x,\tau)\, O(\phi(\tau),\pi(\tau))\, A(\phi(\tau),\pi(\tau)) \times \exp\left\{\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau\int dx\left[i\pi(x,\tau)\partial_\tau\phi(x,\tau) - H[\phi(x,\tau),\pi(x,\tau)]\right]\right\}$$

(Note: from now on we suppress the explicit mention T_τ and assume time-order always).

• The use of path integrals is largely based in the formulas for gaussian integrals over N complex variables (say $\{u_i\}$):

(i) $$\left(\prod_i \int \frac{du_i\, du_i^*}{2i\pi}\right) \exp\left[-\sum_{ij} u_i^* A_{ij} u_j + \sum_i (h_i^* u_i + h_i u_i^*)\right] = \frac{\exp\left[\sum_{ij} h_i^* (A^{-1})_{ij} h_j\right]}{\det(A)}$$

(ii) $$\langle u_i^* u_j\rangle = \frac{\left(\prod_i \int \frac{du_i\, du_i^*}{2i\pi}\right) u_i^* u_j\, e^{-\sum_{ij} u_i^* A_{ij} u_j}}{\left(\prod_i \int \frac{du_i\, du_i^*}{2i\pi}\right) e^{-\sum_{ij} u_i^* A_{ij} u_j}} = A^{-1}_{ij}$$

(iii) When A is diagonal:

$$\langle u^*(q_1)\, u(q_2)\rangle = \frac{\int Du(q)\, u^*(q_1)\, u(q_2)\, e^{-\frac{1}{2}\sum_q A(q) u^*(q) u(q)}}{\int Du(q)\, e^{-\frac{1}{2}\sum_q A(q) u^*(q) u(q)}} = \frac{1}{A(q_1)}\,\delta_{q_1,q_2}$$

- We shall employ these useful expressions pretty much in a moment.

- ϕ-ϕ correlation

- The path-integral formalism is particularly convenient with quadratic Hamiltonians as that of p. 17.

- Let's compute $\langle[\Phi(\vec{r})-\Phi(0)]^2\rangle$, where $\vec{r}\equiv(x,u\tau)$ imaginary time

- Let's first Fourier-Transform (in space and time): $\Phi(\vec{r})=\frac{1}{\beta\Omega}\sum_{\vec{q}}\phi(\vec{q})e^{i\vec{q}\cdot\vec{r}}$

where $\vec{q}\equiv(\kappa,\omega_n/u)$, and $\exp(i\vec{q}\cdot\vec{r})=\exp[i(\kappa x-\omega_n\tau)]$ ($\omega_n\equiv$ Matsubara frequencies)

- Then:

$$\langle[\Phi(\vec{r}_1)-\Phi(\vec{r}_2)]^2\rangle=\frac{1}{(\beta\Omega)^2}\sum_{\vec{q}_1,\vec{q}_2}\langle\phi(\vec{q}_1,\vec{q}_2)\rangle\left[e^{i\vec{q}_1\cdot\vec{r}_1}-e^{i\vec{q}_1\cdot\vec{r}_2}\right]\left[e^{i\vec{q}_2\cdot\vec{r}_1}-e^{i\vec{q}_2\cdot\vec{r}_2}\right]$$

- We must hence evaluate

$$\langle\phi(\vec{q}_1)\phi(\vec{q}_2)\rangle=\frac{1}{Z}\int D\phi(x,\tau)D\theta(x,\tau)\,\phi(\vec{q}_1)\phi(\vec{q}_2)\,e^{-\frac{S}{\hbar}}$$

where (we take $\hbar=1$ from now on):

$$-S\underset{\text{see p. 46}}{=}\int_0^\beta d\tau\int dx\left\{\underbrace{i\frac{1}{\pi}\nabla\theta(x,\tau)\partial_\tau\phi(x,\tau)}_{\text{recall that }\Pi(x,\tau)=\frac{\nabla\theta(x,\tau)}{\pi}\ (\text{p. }13)}-\underbrace{\frac{1}{2\pi}\left[uK(\nabla\theta)^2+\frac{u}{K}(\nabla\phi)^2\right]}_{\text{Hamiltonian of p. }17}\right\}$$

- Fourier-Transforming:

$$\frac{i}{\pi}\nabla\theta\,\partial_\tau\phi=\frac{i}{\pi}\sum_{\vec{q}'}\frac{\theta(\vec{q}')}{\beta\Omega}(+i\kappa')e^{i\vec{q}'\cdot\vec{r}}\sum_{\vec{q}}\frac{\phi(\vec{q})}{\beta\Omega}(-i\omega_n)e^{i\vec{q}\cdot\vec{r}}$$

$$=\frac{i}{\pi}\sum_{\vec{q}}\sum_{\vec{q}'}\frac{\theta(\vec{q}')\phi(\vec{q})}{(\beta\Omega)^2}\kappa'\omega_n\,e^{i(\vec{q}'+\vec{q})\vec{r}}$$

$$\int d\vec{r}\,\frac{i}{\pi}\nabla\theta\,\partial_\tau\phi=\frac{i}{\pi}\sum_{\vec{q}}\sum_{\vec{q}'}\frac{\theta(\vec{q}')\phi(\vec{q})}{\beta\Omega}\kappa'\omega_n\,\delta_{\vec{q}',-\vec{q}}$$

$$=\frac{-i}{\pi\beta\Omega}\sum_{\vec{q}}\kappa\omega_n\,\phi(\vec{q})\theta(-\vec{q})$$

and similarly for the other terms of S, to get:

$$e^{-S} = \exp\left\{\frac{1}{\beta\Omega}\sum_{\vec{q}}\left[-\frac{ik\omega_n}{\pi}\phi(\vec{q})\theta(-\vec{q}) - \frac{uK}{2\pi}k^2\theta(\vec{q})\theta(-\vec{q}) - \frac{u}{2\pi K}k^2\phi(\vec{q})\phi(-\vec{q})\right]\right\}$$

Since for a real field $u(r) \to u(q)^* = u(-q)$, then we may re-write

$$S = \frac{1}{2\beta\Omega}\sum_{\vec{q}}(\theta^*_{\vec{q}}, \phi^*_{\vec{q}})\hat{M}^{-1}\begin{pmatrix}\theta_{\vec{q}}\\ \phi_{\vec{q}}\end{pmatrix} \quad \text{with } \hat{M}^{-1} \equiv \begin{pmatrix}\frac{k^2 uK}{\pi} & \frac{ik\omega_n}{\pi}\\ \frac{ik\omega_n}{\pi} & k^2\frac{u}{K\pi}\end{pmatrix}$$

* Note that now we are interested in the ϕ-ϕ correlation, and hence we may perform the integral over θ. To do so we complete squares in S:

$$e^{-S} = e^{\frac{1}{\beta\Omega}\sum_{\vec{q}}\left\{\frac{1}{2\pi K}\left(\frac{\omega_n^2}{u}+uk^2\right)\phi(-\vec{q})\phi(\vec{q}) - \frac{uK}{2\pi}\tilde{\theta}(-\vec{q})\tilde{\theta}(\vec{q})\right\}}$$

where $\tilde{\theta}(\vec{q}) = \theta(\vec{q}) + \frac{i\omega_n}{uKk}\phi(\vec{q})$

• Using the new variable $\tilde{\theta}$ one has a Gaussian integral on $\tilde{\theta}$, and which actually will appear both in $\int D\phi D\theta\, \phi(\vec{q}_1)\phi(\vec{q}_2) e^{-S/\hbar}$ and in the partition function Z, and hence it will cancel when doing the correlation. We have hence:

$$\langle\phi(\vec{q}_1)\phi(\vec{q}_2)\rangle = \frac{1}{Z_\phi}\int D\phi(x,\tau)\, e^{-S_\phi}\,\phi(\vec{q}_1)\phi(\vec{q}_2)$$

recall that $\phi(q)^* = \phi(-q)$

where $Z_\phi = \int D\phi(x,\tau)\, e^{-S_\phi}$ and

$$S_\phi = \frac{1}{\beta\Omega}\sum_{k,\omega_n}\frac{1}{2\pi K}\left[\frac{\omega_n^2}{u}+uk^2\right]\phi(k,\omega_n)^*\phi(k,\omega_n) =$$

undoing the Fourier-Transform

$$= \frac{1}{2\pi K}\int dx\int_0^\beta d\tau \left\{\underbrace{\frac{1}{u}(\partial_\tau\phi(x,\tau))^2 + u(\partial_x\phi(x,\tau))^2}_{\text{Note the Lorentz-invariant form}}\right\}$$

* Hence

$$\langle \phi(\vec{q}_1)^* \phi(\vec{q}_2) \rangle = \frac{1}{Z_\phi} \int D\phi(x,\tau)\, \phi(\vec{q}_1)^* \phi(\vec{q}_2)\, e^{-\frac{1}{2}\sum_{\vec{q}} \phi(\vec{q})^* \left[\frac{\frac{\omega_n^2}{u} + u k^2}{\beta \Omega \pi K}\right] \phi(\vec{q})}$$

using (iii) in p. 20

$$= \left[\frac{\pi K \Omega \beta}{\frac{\omega_n^2}{u} + u k^2}\right] \delta_{\vec{q}_1, -\vec{q}_2}$$

. Thus:

$$\langle [\phi(\vec{r}) - \phi(0)]^2 \rangle = \frac{1}{\beta\Omega} \sum_{\vec{q}_1} \frac{\pi K}{\frac{\omega_n^2}{u} + u k^2} \left(e^{i\vec{q}_1 \vec{r}_1} - e^{i\vec{q}_1 \cdot \vec{r}_2}\right)\left(e^{-i\vec{q}_1 \vec{r}_1} - e^{-i\vec{q}_1 \vec{r}_2}\right)$$

$$\sum_{\vec{q}} \equiv \sum_{\omega_n} \Omega \int \frac{dk}{2\pi}$$

$$= \frac{1}{\beta\Omega} \sum_{\vec{q}_1} \frac{\pi K}{\frac{\omega_n^2}{u} + u k^2} \left[2 - 2\cos(\vec{q}_1 \cdot (\underbrace{\vec{r}_1}_{(x,u\tau)} - \underbrace{\vec{r}_2}_{0}))\right] =$$

$$= \frac{1}{\beta} \sum_{\omega_n} \int \frac{dk}{2\pi} \frac{2\pi K}{\frac{\omega_n^2}{u} + u k^2} \left[1 - \cos(kx + \omega_n \tau)\right]$$

. Hence:

$$\langle [\phi(\vec{r}) - \phi(0)]^2 \rangle = K \left\{ \frac{1}{\beta\Omega} \sum_{\vec{q}} \frac{2\pi u}{\omega_n^2 + u^2 k^2} \left[1 - \cos(kx + \omega_n \tau)\right] \right\}$$

From now on we denote:

$$\boxed{F_1(\vec{r}) = \frac{1}{\beta\Omega} \sum_{\vec{q}} \frac{2\pi u}{\omega_n^2 + u^2 k^2} \left[1 - \cos(kx + \omega_n \tau)\right]}$$

Hence: $\boxed{\langle [\phi(\vec{r}) - \phi(0)]^2 \rangle = K\, F_1(\vec{r})}$

* <u>θ-θ correlation</u>

. The θ-θ correlation is done exactly the same as the φ-φ correlation. Note that the Hamiltonian in p. 17 is invariant if we exchange $K\theta \to \phi/K$. Hence the θ-θ correlation is the same but changing $K \to 1/K$: $\boxed{\langle [\theta(\vec{r}) - \theta(0)]^2 \rangle = \frac{1}{K} F_1(\vec{r})}$

* Φ-θ correlation

* Since $S = \frac{1}{2} \sum_{\vec q} (\theta^*_{\vec q}, \phi^*_{\vec q}) \frac{\hat M^{-1}}{\beta\Omega} \begin{pmatrix} \theta_{\vec q} \\ \phi_{\vec q} \end{pmatrix}$

Then, from (ii) in p. 20:

$$\langle \phi(\vec q_1)\, \theta(\vec q_2)\rangle = M_{\phi\theta}(\vec q_1)\, \beta\Omega \;\delta_{\vec q_1, -\vec q_2}$$

where the inverse of M^{-1} is given by:

$$\hat M = \frac{\pi}{k^2(u^2k^2+\omega_n^2)} \begin{bmatrix} k^2 \frac{u}{K} & -ik\omega_n \\ -ik\omega_n & k^2 uK \end{bmatrix}$$

and $M_{\phi\theta}(\vec q) = \dfrac{-i\pi k\omega_n}{k^2(u^2k^2+\omega_n^2)}$

Hence $\langle \phi(\vec q_1)\,\theta(\vec q_2)\rangle = \dfrac{-i\pi k\omega_n \beta\Omega}{k^2(u^2k^2+\omega_n^2)}\, \delta_{\vec q_1,-\vec q_2}$

and then:

$$\langle \phi(\vec r_1)\,\theta(\vec r_2)\rangle = \frac{1}{(\beta\Omega)^2} \sum_{\vec q_1,\vec q_2} \langle \phi(\vec q_1)\,\theta(\vec q_2)\rangle\, e^{i\vec q_1\cdot\vec r_1 + i\vec q_2\cdot\vec r_2}$$

$$= -\frac{1}{\beta\Omega} \sum_{\vec q} \frac{i\omega_n k\pi}{k^2(u^2k^2+\omega_n^2)}\, e^{i\vec q\cdot(\vec r_1-\vec r_2)}$$

Let $\boxed{F_2(r) = \frac{1}{\beta\Omega} \sum_{\vec q} e^{i(kx-\omega_n\tau)} \frac{(-i\,2\pi\,\omega_n/\tau)}{\omega_n^2+u^2k^2}}$

Then: $\boxed{\langle \phi(r)\,\theta(0)\rangle = \frac{1}{2} F_2(r)}$

* Note that the integrals over k diverge at large k. One has to impose a momentum cut-off $e^{-\alpha|k|}$.

If we do this, and summing over ω_n $\left(\frac{1}{\beta}\sum_n \to \int \frac{d\omega}{2\pi}\right)$ we obtain (we skip the details):

* For $T=0$:

$$F_1(\vec{r}) = \frac{1}{2}\log\left\{\frac{x^2+(u|\tau|+\alpha)^2}{\alpha^2}\right\}$$

$$F_2(\vec{r}) = -i\,\mathrm{Arg}\left[y_\alpha + ix\right] \quad \text{with } y_\alpha = u\tau + \alpha\,\mathrm{sign}(\tau)$$

* For $T\geq 0$

$$F_1(\vec{r}) = \frac{1}{2}\log\left\{\frac{\beta^2u^2}{\pi^2\alpha^2}\left(\sinh^2\left(\frac{\pi x}{\beta u}\right)+\sin^2\left(\frac{\pi\tau}{\beta}\right)\right)\right\}$$

$$F_2(\vec{r}) = -i\,\mathrm{Arg}\left\{\tan\left(\frac{\pi y_\alpha}{\beta u}\right)+i\tanh\left(\frac{\pi x}{\beta u}\right)\right\}$$

* Correlation functions of the exponential of the fields

· Let's compute now correlations of the form

$$I = \left\langle \prod_j e^{i[A_j\phi(\vec{r}_j)+B_j\theta(\vec{r}_j)]}\right\rangle \qquad \vec{r}_j \equiv (x_j, u\tau_j), \quad A_j, B_j \equiv \text{coefficients}$$

· It's easy to see that only correlations for which $\sum_j A_j = 0$ and $\sum_j B_j = 0$ are different than zero.

The argument is simple. Note that the action just depends on $\nabla\theta$ and $\nabla\phi$ and hence it's identical for ϕ and $\phi + \pi/(\sum_j A_j)$ (and similarly for θ). But upon this transformation $I \to e^{i\pi} I$. Since in the path integral one has to sum over all configurations these two contribution cancel each other. As a result $I = 0$ if $\sum_j A_j \neq 0$ or $\sum_j B_j \neq 0$

- Let's Fourier-transform:

$$A(\vec{q}) = \sum_j A_j \, e^{-i(kx_j - \omega_n \tau_j)}$$

and similarly for B_j. Then:

$$\sum_j \left[A_j \Phi(\vec{r}_j) + B_j \Theta(\vec{r}_j)\right] = \frac{1}{\beta\Omega} \sum_{\vec{q}} \left[A(\vec{q})\Phi(-\vec{q}) + B(\vec{q})\Theta(-\vec{q})\right]$$

Hence, using the form of S of p. 22

$$I = \frac{1}{Z}\int D\Phi\, D\Theta \exp\left\{-\frac{1}{2}\frac{1}{\beta\Omega}\sum_{\vec{q}}\left[(\theta_{-\vec{q}},\Phi_{-\vec{q}})\hat{M}^{-1}\begin{pmatrix}\theta_{\vec{q}}\\ \Phi_{\vec{q}}\end{pmatrix} - i\begin{bmatrix}(B(\vec{q}),A(-\vec{q}))\begin{pmatrix}\theta_{\vec{q}}\\ \Phi_{\vec{q}}\end{pmatrix}\\ +(\theta_{-\vec{q}},\Phi_{-\vec{q}})\begin{pmatrix}B(\vec{q})\\ A(\vec{q})\end{pmatrix}\end{bmatrix}\right]\right\}$$

- We can use now the useful expression (i) in p. 20, where $u = \begin{pmatrix}B\\ A\end{pmatrix}$ to obtain that:

$$I = \exp\left\{-\frac{1}{2}\frac{1}{\beta\Omega}\sum_{\vec{q}}(B(\vec{q}),A(-\vec{q}))\hat{M}\begin{pmatrix}B(\vec{q})\\ A(\vec{q})\end{pmatrix}\right\}$$

where we recall that $\hat{M}$ has the form:

$$\hat{M} = \frac{\pi}{k^2(u^2k^2+\omega_n)}\begin{bmatrix}k^2\frac{u}{K} & -ik\omega_n\\ -ik\omega_n & k^2uK\end{bmatrix}$$

- For example, let's have a look to the AA term in the exponent

$$-\frac{1}{2\beta\Omega}\sum_{\vec{q}} A(-\vec{q})\frac{\pi u K}{u^2k^2+\omega_n}A(\vec{q}) \overset{\text{un-doing the Fourier transform}}{=}$$

$$= -\frac{1}{2\beta\Omega}\sum_{i,j}\sum_{\vec{q}} A_iA_j\, e^{i\vec{q}(\vec{r}_i-\vec{r}_j)}\frac{\pi u K}{u^2k^2+\omega_n} \overset{\text{due to parity}}{=}$$

$$= -\frac{1}{2\beta\Omega}\sum_{i,j}\sum_{\vec{q}} A_iA_j\cos\vec{q}(\vec{r}_i-\vec{r}_j)\frac{\pi u K}{u^2k^2+\omega_n}$$

$$= \underbrace{-\frac{1}{2\beta\Omega}\sum_{ij}\sum_{\vec{q}} A_iA_j\left[\cos\vec{q}(\vec{r}_i-\vec{r}_j)-1\right]\frac{\pi u K}{u^2k^2+\omega_n}}_{\text{Recall that } F_1(\vec{r}) = \frac{1}{\beta\Omega}\sum_{\vec{q}}\frac{2\pi u}{u^2k^2+\omega_n}[1-\cos\vec{q}(\vec{r}_i-\vec{r}_j)] \text{ (p. 23)}} \underbrace{-\frac{1}{2\beta\Omega}\sum_i A_i\sum_j A_j \underbrace{\sum_{\vec{q}}\frac{\pi u K}{u^2k^2+\omega_n}}_{\text{This diverges to } +\infty}}$$

$\Downarrow$

Due to the minus sign this means that I is zero unless $\sum_j A_j = 0$, as we already mentioned before.

- Hence the AA contribution to the exponent reduces to

$$\frac{1}{4}\sum_{ij} A_i A_j K F_1(\bar{r}_i - \bar{r}_j) = \frac{1}{2}\sum_{i<j} A_i A_j K F_1(\bar{r}_i - \bar{r}_j)$$

We can proceed similarly with the BB and AB terms to obtain the final expression

$$\boxed{\langle \prod_j e^{i(A_j\phi(\bar{r}_j) + B_j\theta(\bar{r}_j))}\rangle = e^{-\frac{1}{2}\sum_{i<j}\left[\begin{array}{l}(-A_iA_jK - B_iB_jK^{-1})F_1(\bar{r}_i-\bar{r}_j)\\ +(A_iB_j+B_iA_j)F_2(\bar{r}_i-\bar{r}_j)\end{array}\right]}}$$

- This expression is <u>very useful</u> indeed!

- Recall that (p. 23 and 24):

$$\langle[\phi(\bar{r}_i)-\phi(\bar{r}_j)]^2\rangle = K F_1(\bar{r}_i-\bar{r}_j)$$

$$\langle[\theta(\bar{r}_i)-\theta(\bar{r}_j)]^2\rangle = K^{-1} F_1(\bar{r}_i-\bar{r}_j)$$

$$\langle\phi(\bar{r}_i)\theta(\bar{r}_j)\rangle = \frac{1}{2}F_2(\bar{r}_i-\bar{r}_j)$$

- Hence the exponent in I becomes:

$$+\frac{1}{4}\sum_{i,j}\left\{\begin{array}{l}A_iA_j\langle[\phi(\bar{r}_i)-\phi(\bar{r}_j)]^2\rangle + B_iB_j\langle[\theta(\bar{r}_i)-\theta(\bar{r}_j)]^2\rangle\\ -2A_iB_j\langle\phi(\bar{r}_i)\theta(\bar{r}_j)\rangle - 2B_iA_j\langle\theta(\bar{r}_i)\phi(\bar{r}_j)\rangle\end{array}\right\}$$

$$= -\frac{1}{2}\sum_{i,j}\langle A_iA_j\phi(\bar{r}_i)\phi(\bar{r}_j) + B_iB_j\theta(\bar{r}_i)\theta(\bar{r}_j) + A_iB_j\phi(\bar{r}_i)\theta(\bar{r}_j) + B_iA_j\theta(\bar{r}_i)\phi(\bar{r}_j)\rangle$$

$$+\frac{1}{4}\sum_{ij}A_iA_j\cancel{\langle\phi(\bar{r}_i)^2+\phi(\bar{r}_j)^2\rangle} + \frac{1}{4}\sum_{ij}B_iB_j\cancel{\langle\theta(\bar{r}_i)^2+\theta(\bar{r}_j)^2\rangle}$$

0 ⟶ since $\sum_j A_j = \sum_j B_j = 0$ ⟵ 0

$$= -\frac{1}{2}\langle\left(\sum_i(A_i\phi(\bar{r}_i)+B_i\theta(\bar{r}_i))\right)^2\rangle$$

Hence we obtain the <u>very important expression</u>

$$\boxed{\langle\prod_j e^{i(A_j\phi(\bar{r}_j)+B_j\theta(\bar{r}_j))}\rangle = e^{-\frac{1}{2}\langle\left[\sum_i(A_i\phi(\bar{r}_i)+B_i\theta(\bar{r}_i))\right]^2\rangle}}$$

* Density-density correlations

· Let's have a look first to the density-density correlations.

Recall that:

$$\rho(\vec r) = \rho_R(\vec r) + \rho_L(\vec r) + \psi_R^+(\vec r)\psi_L(\vec r) + \psi_L^+(\vec r)\psi_R(\vec r) \overset{p.\,14}{=}$$

dropping the Klein factors which don't contain space-time dependence.

$$= -\frac{1}{\pi}\nabla\phi(\vec r) + \frac{1}{2\pi\alpha}\left[e^{2ik_Fx}e^{-2i\phi(\vec r)} + e^{-2ik_Fx}e^{2i\phi(\vec r)}\right]$$

· Then

$$\langle\rho(\vec r)\rho(0)\rangle = \frac{1}{\pi^2}\langle\nabla\phi(\vec r)\nabla\phi(0)\rangle + \frac{1}{2\pi\alpha^2}\left\{\begin{array}{l} e^{2ik_Fx}\left[\langle e^{-2i(\phi(\vec r)-\phi(0))}\rangle + \langle e^{-2i(\phi(\vec r)+\phi(0))}\rangle\right] \\ +e^{-2ik_Fx}\left[\langle e^{2i(\phi(\vec r)-\phi(0))}\rangle + \langle e^{2i(\phi(\vec r)+\phi(0))}\rangle\right]\end{array}\right\}$$

$$-\frac{1}{2\pi^2\alpha}\langle\nabla\phi(\vec r)\left[e^{-2i\phi(0)} + e^{2i\phi(0)}\right]\rangle - \frac{1}{2\pi^2\alpha}\langle\left[e^{-2i\phi(\vec r)} + e^{2i\phi(\vec r)}\right]\nabla\phi(0)\rangle$$

· Let's have a look to the different terms:

$$\langle\nabla\phi(\vec r)\nabla\phi(0)\rangle = -\frac{1}{(\beta\Omega)^2}\sum_{\vec q}\sum_{\vec q'} kk'\langle\phi(\vec q)\phi(\vec q')\rangle e^{i\vec q\vec r} \overset{p\,23}{=}$$

$$= -\frac{1}{(\beta\Omega)^2}\sum_{\vec q}\frac{\pi K\Omega\beta}{\frac{\omega_n^2}{u}+uk^2}(-k^2)e^{i\vec q\cdot\vec r} = \frac{\pi K}{\beta\Omega}\sum_{\vec q}\frac{k^2e^{i\vec q\cdot\vec r}}{u\left(\left(\frac{\omega_n}{u}\right)^2+k^2\right)} =$$

$$= \pi K\iint\frac{dk\,d(\omega/u)}{(2\pi)^2}\frac{k^2e^{i\left(kx-\frac{\omega}{u}(\tau u)\right)}}{\left(\frac{\omega}{u}\right)^2+k^2} \overset{y=\tau u}{=} \frac{\pi K}{(2\pi)^2}\int_{-\infty}^{\infty}dk\,k^2e^{ikx}\int_{-\infty}^{\infty}d\omega\frac{e^{-i\omega y}}{(\omega^2+k^2)}$$

$$= \frac{\pi K}{(2\pi)^2}\int_0^{\infty}dk\,k^2(e^{ikx}+e^{-ikx})\int_{-\infty}^{\infty}\frac{d\omega\,e^{-i\omega y}}{\omega^2+k^2} = \frac{\pi^2K}{(2\pi)^2}\int_0^{\infty}dk\,k\,(e^{ikx}+e^{-ikx})e^{-k|y|} =$$

We employ $\int_{-\infty}^{\infty}d\omega\,\frac{k\,e^{-i\omega y}}{\omega^2+k^2} = \pi e^{-k|y|}$

$$= \frac{\pi^2K}{(2\pi)^2}\int_{-\infty}^{\infty}dk\,|k|\,e^{-|y||k|}e^{ikx} = \frac{\pi^2K}{(2\pi)^2}\,2\left(\frac{-\partial}{\partial|y|}\right)\left[\frac{|y|}{|y|^2+x^2}\right] = \frac{K}{2}\frac{|y|^2-x^2}{(|y|^2+x^2)^2}$$

$$* \ \langle e^{-2i(\phi(\vec r)-\phi(0))}\rangle \overset{p.\,27}{=} e^{-2\langle(\phi(\vec r)-\phi(0))^2\rangle} \overset{p.\,23}{=} e^{-2KF_1(\vec r)} \overset{p.\,25}{=}$$

$$= e^{-K\log\left[\frac{r^2}{\alpha^2}\right]} = \left(\frac{r}{\alpha}\right)^{-2K}$$

$$* \ \langle e^{-2i(\phi(\vec r)+\phi(0))}\rangle = 0 \quad \left(\text{since } \sum_j A_j \neq 0\right)$$

* Finally

$$\langle \nabla\phi(r)\, e^{2i\phi(0)}\rangle = \frac{i}{\beta\Omega}\sum_{q} K \underbrace{\langle \Phi(q)\, e^{\frac{2i}{\beta\Omega}\sum_{q'}\phi(q')}\rangle}_{=\langle \Phi(-q)\, e^{\frac{2i}{\beta\Omega}\sum_{q'}\phi(q')}\rangle} = 0$$

we introduce the cut-off in y

$y \to y_\alpha = (u\tau) + \alpha\, \text{sign}(\tau)$

* Hence

$$\langle \rho(r)\,\rho(0)\rangle = \frac{K}{2\pi^2}\left[\frac{y_\alpha^2 - x^2}{(y_\alpha^2+x^2)^2}\right] + \frac{1}{2\pi\alpha^2}\left\{e^{2ik_F x} + e^{-2ik_F x}\right\}\left(\frac{r}{\alpha}\right)^{-2K}$$

$$\boxed{\langle \rho(r)\,\rho(0)\rangle = \underbrace{\frac{K}{2\pi^2}\left[\frac{y_\alpha^2 - x^2}{(y_\alpha^2+x^2)^2}\right]} + \underbrace{\frac{2}{2\pi\alpha^2}\cos 2k_F x \left(\frac{\alpha}{r}\right)^{2K}}}$$

$q \sim 0$ term → It decays as $1/x^2$ as expected for free fermion correlations
(The Fourier transform of this gives the compressibility back)

$2k_F x$ part

It behaves as a non-universal power-law with an exponent dependent on interactions. A system with these anomalous power-law correlations is referred to as a

Luttinger Liquid

* Note: strictly speaking we should take $\alpha \to 0$, but $\alpha \to 0$ leads to ultraviolet divergences. In lattice model α is given by the lattice spacing. In continuous models there's no small length scale acting as cut-off, but one can show that a finite range in the interactions will play a similar role. So we will keep a finite cut-off α in the following.

* Pairing correlations

* Let's calculate now other type of correlation.

The operator $O_{SU}(r) = \psi^\dagger(r)\,\psi^\dagger(r+a)$ describes the tendency to pairing. Note that one should create the two fermions at neighboring sites ($a \equiv$ lattice constant) (then we may tend $a \to 0$).

* let's have a look to the correlations of this operator.

$$O_{SU}(r) = \underbrace{\psi_R^\dagger(r)\,\psi_R^\dagger(r+a) + \psi_L^\dagger(r)\,\psi_L^\dagger(r+a)} + \psi_R^\dagger(r)\,\psi_L^\dagger(r+a) + \psi_L^\dagger(r)\,\psi_R^\dagger(r+a)$$

Note that these terms must be largely suppressed due to Pauli-blocking when $a \to 0$

- Let's write $O_{SU}(r)$ in terms of the θ and ϕ fields. From p. (14) we obtain (removing the Klein factors)

$$\psi_R^\dagger(r)\,\psi_R^\dagger(r+a) \underset{a\to 0}{=} \frac{1}{2\pi\alpha}\, e^{-2ik_F x}\, e^{+2i(\phi(r)-\theta(r))}$$

$$\psi_L^\dagger(r)\,\psi_L^\dagger(r+a) = \frac{1}{2\pi\alpha}\, e^{+2ik_F x}\, e^{+2i(-\phi(r)-\theta(r))}$$

$$\psi_R^\dagger(r)\,\psi_L^\dagger(r+a) = \frac{1}{2\pi\alpha}\, e^{-2i\theta(r)} = \psi_L^\dagger(r)\,\psi_R^\dagger(r+a)$$

Hence

$$O_{SU}(r) = \frac{2}{2\pi\alpha}\left\{ e^{-2i\theta(r)}\cos\left[2\phi(r)-2k_F x\right] + e^{-2i\theta(r)}\right\}$$

* Recall from our previous discussion that each correlation of an exponential of a field (θ or ϕ) leads to a power decay. Note that the first term in $O_{SU}(r)$ hence will contribute with a faster decay as a function of space or time. These reflects our intuitive idea that the $\psi_R^\dagger\psi_R^\dagger + \psi_L^\dagger\psi_L^\dagger$ term decays strongly due to Pauli-blocking.

* Hence $O_{SU}(r) \simeq \dfrac{1}{\pi\alpha}\, e^{-2i\theta(r)}$

- Let's have a look to the correlation:

$$\langle O_{SU}(r)\, O_{SU}^\dagger(0)\rangle \cong \frac{1}{(\pi\alpha)^2}\langle e^{-2i(\theta(r)-\theta(0))}\rangle = \frac{1}{(\pi\alpha)^2}\, e^{-2\langle(\theta(r)-\theta(0))^2\rangle}$$

$$= \frac{1}{(\pi\alpha)^2}\, e^{-\frac{2}{K}F_1(r)} = \frac{1}{(\pi\alpha)^2}\, e^{-\frac{1}{K}\log\left(\frac{r^2}{\alpha^2}\right)} = \frac{1}{(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{2/K}$$

Hence $\boxed{\langle O_{SU}(r)\, O_{SU}^\dagger(0)\rangle = \dfrac{1}{(\pi\alpha)^2}\left(\dfrac{\alpha}{r}\right)^{2/K}}$

* Note that the superconducting correlations also decay as a power-law with a non-universal exponent.

- <u>Note:</u> that if K is larger, $\langle O_{SU}(r)\, O_{SU}^\dagger(0)\rangle$ decays slower, and hence the tendency for superconducting fluctuations is stronger (this is clear because $K>1$ means attractive interactions (p. (17))). However when K decreases density fluctuations are stronger (recall that for $K<1$, we have repulsive interactions) p. (17)

• "Phase diagram" for the spin-less system

• From the various correlation functions $R(r)$ we can define the susceptibilities:

$$\chi(k,\omega_n)=\int_0^\beta d\tau \int dx \; R(x,\tau)\, e^{-i(kx-\omega_n\tau)}$$

• Let's do a scaling analysis. If (as we saw above) R decays as a power law $R(r)\sim r^{-\nu}$ then (for $\beta\to\infty$)

$$\chi(k,\omega_n)\sim \max[\delta k,\omega_n]^{\nu-2}$$

where $\delta k = \cancel{k}-2k_F$ for density correlations (this is because for $\rho\rho$ correlations $R_{\rho\rho}(r)\sim e^{2ik_F x}\, r^{-\nu}$ (p. 29))
$= k$ for pairing correlations

Note: $\chi(k,\omega_n)\sim \int_0^\infty d\tau \int_{-\infty}^{\infty} dx \; \frac{1}{r^\nu}\, e^{-i\vec{q}\cdot\vec{r}} \sim \int \frac{r\,dr}{r^\nu}\, e^{-i\vec{q}\cdot\vec{r}} \sim q^{\nu-2}$

• Note that when $\nu-2<0$ the susceptibilities become divergent.
Such a divergent susceptibility indicates that any coupling to the corresponding operator, however weak, would induce a finite response. Such a weak coupling could, for example, be provided by the presence of other 1D chains forming a 3D network. A divergent susceptibility indicates the state would like to order into, if it were not prevented by its 1D nature.

• So let's use the susceptibilities to build a sort of "phase diagram". of the 1D system, i.e. we determine the most divergent fluctuation of the 1D system, i.e. that more likely to be stabilized by a very weak 3D coupling).

• For $K<1$ (repulsive interactions)
$\rho\rho \longrightarrow r^{-2K} \longrightarrow \nu=2K<2 \longleftarrow$ density-~~density~~ modulations dominate
$O_{su}O_{su}^\dagger \longrightarrow r^{-2/K} \longrightarrow \nu=\frac{2}{K}>2$

The corresponding ordered state would be a periodic modulation of the density (with wavevector $2k_F$, recall p. 29). This would be a so-called Charge-density wave (CDW)

• We can thus view the 1D system with $K<1$ as a CDW whose perfect order is destroyed by quantum fluctuations.

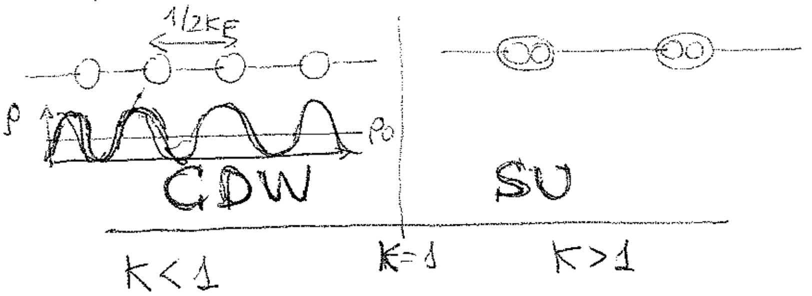
(Note: in a classical picture $\rho(x) \sim \cos(2k_F x - 2\phi)$ where ϕ would be the phase of the density modulation. In 1D ϕ fluctuates killing the CDW ordering.)

- For $K>1$ (attractive interactions) $\rightarrow O_{SU} O_{SU}^{\dagger}$ commute $\rightarrow$ the system has a diverging susceptibility towards pair ordering.

$\rightarrow O_{SU} \sim e^{-2i\theta} \Rightarrow$ the superconducting phase θ tends to acquire a constant value (in 1D θ fluctuates and this ordering is again killed).

- Note that when the superconducting phase "likes" to order, the density fluctuation increase and viceversa. There's a duality between superconducting phase and charge given by the commutations between ϕ and θ.

- So, we have finally a sort of "phase diagram" for spin-less fermions:



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* Momentum distribution

- Another important correlation is the single-particle green's function. For right movers:

$$\langle G_R(r)\rangle = -\langle \psi_R(r)\psi_R^{\dagger}(0)\rangle = -\frac{e^{ik_F x}}{2\pi\alpha}\langle e^{i(\phi(r)-\theta(r))}e^{-i(\phi(0)-\theta(0))}\rangle$$

$$= \frac{-e^{ik_F x}}{2\pi\alpha}\exp\left\{-\left[\frac{K+K^{-1}}{2}F_1(r)+F_2(r)\right]\right\} \longrightarrow \text{which is again a power-law.}$$

Note that the power law decay $r^{-\frac{(K+K^{-1})}{2}}$ is always faster than r^{-2} for all $K\neq 1$ (i.e. faster than for free fermions). Single-particle excitations are really not welcome in 1D, and interactions just make it worse (since it goes as $K+\frac{1}{K}$ it's irrelevant whether the interactions are attractive or repulsive!).

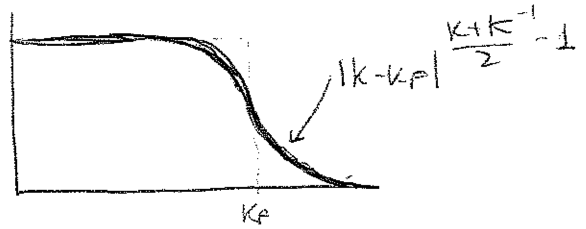
· We may obtain the momentum distribution

$$n(k) = \int dx\, e^{-ikx}\, G_R(x, \tau = 0^+) \stackrel{\text{at } T=0}{=}$$

$$= \int dx\, e^{i(k_F - k)x} \left(\frac{-1}{2\pi\alpha}\right) \left(\frac{\alpha}{\sqrt{x^2+\alpha^2}}\right)^{\frac{K+K^{-1}}{2}} e^{i\,\mathrm{Arg}(-\alpha + ix)}$$

$$\sim |k - k_F|^{\frac{K+K^{-1}}{2} - 1}$$

· The momentum distribution looks like this:



Instead of a discontinuity at k_F one finds an essential power-law singularity

Recall from p.① that the discontinuity at k_F signals in a Fermi liquid that fermionic quasiparticles are sharp excitations. Hence individual quasiparticles can't survive in 1D as we already mentioned in p.②.