• THE BASICS OF BOSONIZATION II: SPIN-FULL FERMIONS

SPIN AND CHARGE

- Let's consider now the case of fermions with spin $1/2 = \{\uparrow, \downarrow\}$
- For each species we can employ the boson representation and introduce two sets of fields $(\phi_\uparrow, \theta_\uparrow)$ and $(\phi_\downarrow, \theta_\downarrow)$

Note: in order to ensure proper anticommutation we need also two Klein factors ($U_{\uparrow\downarrow}$)

- The kinetic energy is simply

$$H_{kin} = H_\uparrow^0 + H_\downarrow^0$$

where $$H^0_{\uparrow,\downarrow} = \frac{1}{2\pi}\int dx\, v_F \left[(\nabla\theta_{\uparrow\downarrow}(x))^2 + (\nabla\phi_{\uparrow\downarrow}(x))^2 \right]$$

- Let's have a look now to the interactions. Let's see first the g_4 and g_2 processes (recall p. (15)). Later on we will see the g_1 processes (which contrary to the spin-less case now will play a key role).
- Recall that the g_4 processes occur at the same side of the Fermi surface:

$$\hat{H}_4 = \int dx \sum_{r=R,L} \sum_{\sigma=\uparrow,\downarrow} \left\{ \underbrace{\frac{g_{4\parallel}}{2} \rho_{r,\sigma}(x)\rho_{r,\sigma}(x)}_{\substack{\text{Interaction between}\\ \text{particles with the same}\\ \text{spin} \to g_{4\parallel}}} + \underbrace{\frac{g_{4\perp}}{2} \rho_{r,\sigma}(x)\rho_{r,-\sigma}(x)}_{\substack{\text{Interaction between}\\ \text{particles with opposite}\\ \text{spin} \to g_{4\perp}}} \right\}$$

- Recall that the g_2 processes occur between particles at opposite sides of the Fermi surface, but after the process the particles remain at their sides:

$$\hat{H}_2 = \int dx \sum_{\sigma=\uparrow\downarrow} \left\{ \underbrace{g_{2\parallel}\, \rho_{R,\sigma}(x)\rho_{L,\sigma}(x)}_{\substack{\text{Interaction between}\\ \text{particles with the same}\\ \text{spin} \to g_{2\parallel}}} + \underbrace{g_{2\perp}\, \rho_{R,\sigma}(x)\rho_{L,-\sigma}(x)}_{\substack{\text{Interaction between}\\ \text{particles with opposite}\\ \text{spin} \to g_{2\perp}}} \right\}$$

- Recall from p. (13) that

$$\rho_{R,\uparrow\downarrow}(x) = \frac{1}{2\pi}\left[\nabla\theta_{\uparrow\downarrow}(x) - \nabla\phi_{\uparrow\downarrow}(x)\right]$$

$$\rho_{L,\uparrow\downarrow}(x) = \frac{-1}{2\pi}\left[\nabla\theta_{\uparrow\downarrow}(x) + \nabla\phi_{\uparrow\downarrow}(x)\right]$$

and hence $H_{kin} + H_4 + H_2$ may be reexpressed ~~only~~ as a quadratic form of $\nabla\phi_{\uparrow\downarrow}$ and $\nabla\theta_{\uparrow\downarrow}$.

* $H_{kin} + H_4 + H_2$ remains quadratic but it's clearly not diagonal in the spin index, due to the $g_{2\perp}$ and $g_{4,\perp}$ terms. To diagonalize the Hamiltonian we introduce:

$$\rho(x) = \frac{1}{\sqrt{2}}\left[\rho_\uparrow(x) + \rho_\downarrow(x)\right] \longrightarrow \underline{\text{Charge}}$$

$$\sigma(x) = \frac{1}{\sqrt{2}}\left[\rho_\uparrow(x) - \rho_\downarrow(x)\right] \longrightarrow \underline{\text{Spin}}$$

* This is a unitary transformation for the boson operators and allows to introduce the new boson fields:

$$\boxed{\begin{aligned} \phi_\rho(x) &= \frac{1}{\sqrt{2}}\left[\phi_\uparrow(x) + \phi_\downarrow(x)\right] \;;\; \theta_\rho(x) = \frac{1}{\sqrt{2}}\left[\theta_\uparrow(x) + \theta_\downarrow(x)\right] \\ \phi_\sigma(x) &= \frac{1}{\sqrt{2}}\left[\phi_\uparrow(x) - \phi_\downarrow(x)\right] \;;\; \theta_\sigma(x) = \frac{1}{\sqrt{2}}\left[\theta_\uparrow(x) - \theta_\downarrow(x)\right] \end{aligned}}$$

* It's easy to see that the ρ and σ fields commute whereas (ϕ_ρ, θ_ρ) and $(\phi_\sigma, \theta_\sigma)$ obey the standard commutation relations (p. 13)

$$\left[\phi_\rho(x_1), \theta_\rho(x_2)\right] = i\frac{\pi}{2}\,\mathrm{sign}(x_2 - x_1) \quad \text{and same for } \sigma.$$

* As in p. 14 we can express the single-particle fermion operators as:

$$\boxed{\psi_{r,\sigma}(x) = \frac{1}{\sqrt{2\pi\alpha}}\, U_{r,\sigma}\, e^{irk_F x}\, e^{-\frac{i}{\sqrt{2}}\left[r\phi_\rho(x) - \theta_\rho(x) + \sigma(r\phi_\sigma(x) - \theta_\sigma(x))\right]}}$$

* THE HAMILTONIAN: SPIN-CHARGE SEPARATION

* We may easily re-express H_{kin}, H_4 and H_2 with the ρ, σ fields:

* $H_{kin} = H^0_\rho + H^0_\sigma$

* $H_4 = \frac{1}{4\pi^2}\int dx \left\{\left[g_{4\parallel} + g_{4\perp}\right]\left[(\nabla\phi_\rho(x))^2 + (\nabla\theta_\rho(x))^2\right] + \left[g_{4\parallel} - g_{4\perp}\right]\left[(\nabla\phi_\sigma(x))^2 + (\nabla\theta_\sigma(x))^2\right]\right\}$

* $H_2 = \frac{1}{4\pi^2}\int dx \left\{\left[g_{2\parallel} + g_{2\perp}\right]\left[(\nabla\phi_\rho(x))^2 - (\nabla\theta_\rho(x))^2\right] + \left[g_{2\parallel} - g_{2\perp}\right]\left[(\nabla\phi_\sigma(x))^2 - (\nabla\theta_\sigma(x))^2\right]\right\}$

* The $\underline{g_\perp \text{ processes}}$ demand a little bit more care. Recall from p. 15 that the g_1 processes correspond to backscatterings where fermions exchange sides. For fermions of equal spin ($g_{1\parallel}$) that's not an issue (and $g_{1\parallel}$ contributes to $g_{2\parallel}$), but for fermions of opposite spins it's different.

$$\hat{H}_1 = \int dx\, g_{1\parallel} \sum_\sigma \psi^+_{L\sigma} \psi^+_{R\sigma} \psi_{L\sigma} \psi_{R\sigma} + \int dx\, g_{1\perp} \sum_\sigma \psi^+_{L\sigma} \psi^+_{R\sigma} \psi_{L-\sigma} \psi_{R\sigma}$$

$$-\int dx\, g_{1\parallel} \sum_\sigma (\psi^+_{L\sigma}\psi_{L\sigma})(\psi^+_{R\sigma}\psi_{R\sigma})$$

$$-\int dx\, g_{1\parallel} \sum_\sigma \rho_{L\sigma}\rho_{R\sigma}$$

Identical to a $g_{2\parallel}$ process as we mentioned already

If we redefine $\tilde{g}_{2\parallel} \equiv g_{2\parallel} - g_{1\parallel}$ we may reabsorb these terms in the $g_{2\parallel}$ processes

$$\hat{H}_{1\perp} = \int dx\, g_{1\perp} \sum_\sigma (\psi^+_{L\sigma}\psi_{R\sigma})(\psi^+_{R-\sigma}\psi_{L-\sigma})$$

$$\psi^+_{L\sigma}\psi_{R\sigma} = \frac{1}{2\pi\alpha} e^{-\frac{i}{\sqrt{2}}[\phi_\rho + \theta_\rho + \sigma(\phi_\sigma + \theta_\sigma)]} \times e^{-\frac{i}{\sqrt{2}}[\phi_\rho - \theta_\rho + \sigma(\phi_\sigma - \theta_\sigma)]}$$

$$= \frac{1}{2\pi\alpha} e^{i\sqrt{2}[\phi_\rho + \sigma\phi_\sigma]}$$

$$\psi^+_{R-\sigma}\psi_{L-\sigma} = \frac{1}{2\pi\alpha} e^{-i\sqrt{2}[\phi_\rho - \sigma\phi_\sigma]}$$

Hence $\hat{H}_{1\perp} = \frac{1}{(2\pi\alpha)^2} \int dx\, g_{1\perp} \sum_\sigma e^{i2\sqrt{2}\sigma\phi_\sigma}$

$$\Rightarrow \boxed{\hat{H}_{1\perp} = \int dx\, \frac{2g_{1\perp}}{(2\pi\alpha)^2} \cos(2\sqrt{2}\phi_\sigma)}$$

Note that these terms have a more involved form than the more $H_{kin} + H_4 + H_2 + H_{1\parallel}$. In particular the Hamiltonian is not anymore Harmonic.

- But note that something remarkable occurs. The Hamiltonian splits clearly into two separate parts, one for the density (charge part $\Rightarrow \hat{H}_\rho$) and one for the spin ($\hat{H}_\sigma$)

$$\hat{H} = \hat{H}_\rho + \hat{H}_\sigma$$

where the charge part is:

$$\hat{H}_\rho = \frac{1}{2\pi}\int dx\, v_F [(\nabla\theta_\rho)^2 + (\nabla\phi_\rho)^2]$$
$$+ \frac{1}{4\pi^2}\int dx\, (g_{4\parallel} + g_{4\perp})[(\nabla\phi_\rho)^2 + (\nabla\theta_\rho)^2]$$
$$+ \frac{1}{4\pi^2}\int dx\, (\tilde{g}_{2\parallel} + g_{2\perp})[(\nabla\phi_\rho)^2 - (\nabla\theta_\rho)^2]$$
$$= \frac{1}{2\pi}\int dx \left\{ \left[v_F + \frac{(g_{4\parallel}+g_{4\perp})}{2\pi} - \frac{(\tilde{g}_{2\parallel}+g_{2\perp})}{2\pi}\right](\nabla\theta_\rho)^2 + \left[v_F + \frac{(g_{4\parallel}+g_{4\perp})}{2\pi} + \frac{(\tilde{g}_{2\parallel}+g_{2\perp})}{2\pi}\right](\nabla\phi_\rho)^2 \right\}$$

Hence $\hat{H}_\rho$ is harmonic of the form:

$$\boxed{\hat{H}_\rho = \frac{1}{2\pi}\int dx \left[u_\rho K_\rho (\nabla\theta_\rho)^2 + \frac{u_\rho}{K_\rho}(\nabla\phi_\rho)^2\right]}$$

i.e. exactly as for the spin-less case (recall p. 17). The charge velocity (u_ρ) and charge Luttinger parameter (K_ρ) are:

$$\boxed{\begin{aligned} u_\rho &= \left[\left[v_F + \left(\frac{g_{4\parallel}+g_{4\perp}}{2\pi}\right)\right]^2 - \left[\frac{\tilde{g}_{2\parallel}+g_{2\perp}}{2\pi}\right]^2\right]^{1/2} \\ K_\rho &= \left[\frac{v_F + \left(\frac{g_{4\parallel}+g_{4\perp}}{2\pi}\right) - \left(\frac{\tilde{g}_{2\parallel}+g_{2\perp}}{2\pi}\right)}{v_F + \left(\frac{g_{4\parallel}+g_{4\perp}}{2\pi}\right) + \left(\frac{\tilde{g}_{2\parallel}+g_{2\perp}}{2\pi}\right)}\right]^{1/2} \end{aligned}}$$

[Note: for spin-indep. interact. and all g equal $K_\rho \sim 1 - g/2\pi v_F$ and hence $K_\rho < 1$ means $g>0 \to$ repulsion.]

Recall that $\tilde{g}_{2\parallel} = g_{2\parallel} - g_{1\parallel}$

- The spin part is:

$$\begin{aligned} \hat{H}_\sigma = {} & \frac{1}{2\pi}\int dx\, v_F\left[(\nabla\theta_\sigma)^2 + (\nabla\phi_\sigma)^2\right] \\ & + \frac{1}{4\pi^2}\int dx\, (g_{4\parallel} - g_{4\perp})\left[(\nabla\theta_\sigma)^2 + (\nabla\phi_\sigma)^2\right] \\ & + \frac{1}{4\pi^2}\int dx\, (\tilde{g}_{2\parallel} - g_{2\perp})\left[-(\nabla\theta_\sigma)^2 + (\nabla\phi_\sigma)^2\right] \\ & + \int dx\, \frac{2g_{1\perp}}{(2\pi\alpha)^2}\cos(2\sqrt{2}\,\phi_\sigma) \\ = {} & \frac{1}{2\pi}\int dx \left\{\left[v_F + \left(\frac{g_{4\parallel}-g_{4\perp}}{2\pi}\right) - \left(\frac{\tilde{g}_{2\parallel}-g_{2\perp}}{2\pi}\right)\right](\nabla\theta_\sigma)^2 \right. \\ & \left. + \left[v_F + \left(\frac{g_{4\parallel}-g_{4\perp}}{2\pi}\right) + \left(\frac{\tilde{g}_{2\parallel}-g_{2\perp}}{2\pi}\right)\right](\nabla\phi_\sigma)^2\right\} \\ & + \frac{2}{(2\pi\alpha)^2}\int dx\, g_{1\perp}\cos(2\sqrt{2}\,\phi_\sigma) \end{aligned}$$

We may define the spin velocity (u_σ) and spin Luttinger parameter (K_σ)

$$\boxed{\begin{aligned} u_\sigma &= \left[\left[v_F + \left(\frac{g_{4\parallel}-g_{4\perp}}{2\pi}\right)\right]^2 - \left[\frac{\tilde{g}_{2\parallel}-g_{2\perp}}{2\pi}\right]^2\right]^{1/2} \\ K_\sigma &= \left[\frac{v_F + \left(\frac{g_{4\parallel}-g_{4\perp}}{2\pi}\right) - \left(\frac{\tilde{g}_{2\parallel}-g_{2\perp}}{2\pi}\right)}{v_F + \left(\frac{g_{4\parallel}-g_{4\perp}}{2\pi}\right) + \left(\frac{\tilde{g}_{2\parallel}-g_{2\perp}}{2\pi}\right)}\right]^{1/2} \end{aligned}}$$

[Note: for spin-indep. interact. (and all g equal) $K_\sigma \sim 1 + g/2\pi v_F$ and hence $K_\sigma < 1$ means $g<0 \to$ attraction]

$\tilde{g}_{2\parallel} = g_{2\parallel} - g_{1\parallel}$

and we obtain the final form of the spin Hamiltonian:

$$\boxed{\hat{H}_\sigma = \frac{1}{2\pi}\int dx\left[u_\sigma K_\sigma(\nabla\theta_\sigma)^2 + \frac{u_\sigma}{K_\sigma}(\nabla\phi_\sigma)^2\right] + \frac{2g_{1\perp}}{(2\pi\alpha)^2}\int dx\cos(2\sqrt{2}\,\phi_\sigma)}$$

→ this Hamiltonian is not any more Harmonic, but a so-called sine-Gordon Hamiltonian

• Hence there's a complete separation between density and spin.
This is known as spin-charge separation
(Note: this of course precludes any kind of single-particle excitation which will carry both charge and spin.)

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* Physical observables

- Let's have a look to different physical observables which will characterize the spin-full system.

• Compressibility

- We may proceed as in p. 18 to calculate the compressibility.
The term with the chemical potential in the Hamiltonian is of the form: $-\mu\int dx\,(\rho_\uparrow(x)+\rho_\downarrow(x)) = -\mu\int dx\,[(\rho_{R\uparrow}+\rho_{L\uparrow})+(\rho_{R\downarrow}+\rho_{L\downarrow})]$

$$= +\frac{\mu}{\pi}\int dx\,[\nabla\phi_\uparrow+\nabla\phi_\downarrow] = \frac{\mu\sqrt{2}}{\pi}\int dx\,(\nabla\phi_\rho)$$

Hence with this term the H_ρ Hamiltonian goes as:

$$\frac{1}{2\pi}\int dx\left\{(K_\rho u_\rho)(\nabla\theta_\rho)^2+\left(\frac{u_\rho}{K_\rho}\right)(\nabla\phi_\rho)^2+2\mu\sqrt{2}\,(\nabla\phi_\rho)\right\}$$

$$=\frac{1}{2\pi}\int dx\left\{(K_\rho u_\rho)(\nabla\theta_\rho)^2+\frac{u_\rho}{K_\rho}\left[\nabla\phi_\rho+\frac{\mu\sqrt{2}K_\rho}{u_\rho}\right]^2\right\}+\text{const.}$$

Let $\tilde{\phi}_\rho = \phi_\rho + \frac{\mu\sqrt{2}K_\rho}{u_\rho}x$ → the mean density is $\frac{1}{L}\int dx\,(\rho_\uparrow+\rho_\downarrow) =$

$$= -\frac{\sqrt{2}}{\pi L}\int dx\,(\nabla\phi_\rho) = -\frac{\sqrt{2}}{\pi L}\underbrace{\int dx\,\nabla\tilde{\phi}_\rho}_{\langle\nabla\tilde{\phi}_\rho\rangle=0} + \frac{\sqrt{2}}{\pi L}\cdot\frac{\mu\sqrt{2}K_\rho}{u_\rho}\int dx\,x = \frac{2K_\rho}{\pi u_\rho}\mu$$

Hence $\kappa_\rho = \frac{\partial}{\partial\mu}(\text{mean density}) \longrightarrow \boxed{\kappa_\rho = \frac{2K_\rho}{\pi u_\rho}}$

- Uniform magnetic susceptibility $\left[\dfrac{\chi}{2}_\sigma = \dfrac{\partial}{\partial h} M \text{, with } M = \dfrac{1}{L}\int dx \left(\dfrac{\rho_\uparrow - \rho_\downarrow}{2}\right)\right]$ (39)

- We may proceed in the same way as for the κ_ρ.
- We introduce the effects of a magnetic field (h) in the Hamiltonian:

$$- h \int dx \, \frac{1}{2}(\rho_\uparrow - \rho_\downarrow) = - h \int dx \, \frac{1}{2}\left[(\rho_{R\uparrow} + \rho_{L\uparrow}) - (\rho_{R\downarrow} + \rho_{L\downarrow})\right]$$

$$= + \frac{h}{2\pi}\int dx \, (\nabla\phi_\uparrow - \nabla\phi_\downarrow) = \frac{h}{\sqrt{2}\pi}\int \nabla\phi_\sigma \, dx$$

- For the moment we will assume $g_{1\perp} = 0$ (we will see later that the $g_{1\perp}$ process actually are quite relevant but here we'll keep for the moment simple). Then

$$H_\sigma \longrightarrow H_\sigma + \frac{h}{\sqrt{2}\pi}\int \nabla\phi_\sigma \, dx = \frac{1}{2\pi}\int dx \left\{ (K_\sigma u_\sigma)(\nabla\theta_\sigma)^2 + \left(\frac{u_\sigma}{K_\sigma}\right)(\nabla\phi_\sigma)^2 + \sqrt{2}\, h (\nabla\phi_\sigma)\right\}$$

$$= \frac{1}{2\pi}\int dx \left\{ (K_\sigma u_\sigma)(\nabla\theta_\sigma)^2 + \left(\frac{u_\sigma}{K_\sigma}\right)(\nabla\tilde{\phi}_\sigma)^2 \right\} + \text{constant}$$

where $\tilde{\phi}_\sigma = \phi_\sigma + \dfrac{h K_\sigma}{\sqrt{2}\, u_\sigma} x$

Then: $M = \dfrac{1}{L} \cdot \dfrac{(-1)}{\sqrt{2}\pi}\int \nabla\phi_\sigma \, dx = \dfrac{1}{L \cdot \sqrt{2}\pi}\,\dfrac{h K_\sigma}{\sqrt{2}\, u_\sigma} L = \dfrac{K_\sigma}{2\pi u_\sigma} h$

Hence $\boxed{\kappa_\sigma = \dfrac{K_\sigma}{2\pi u_\sigma}}$

- So (in the case $g_{1\perp} = 0$) one has both a constant compressibility (κ_ρ) and a constant magnetic susceptibility (κ_σ).
- As for the spin-less case the effects of interactions are more relevant in the $2k_F$ terms which appear in various correlation functions. Let's see now the most important observables and their correlations.

- Charge-density and spin density correlations: charge- and spin-density wave

- Let's introduce the operators

- $O_\rho(x) = \sum_{\sigma,\sigma'} \psi^\dagger_\sigma(x)\, \delta_{\sigma\sigma'}\, \psi_{\sigma'}(x) \longrightarrow$ total density and:

- $O^a_\sigma(x) = \sum_{\sigma,\sigma'} \psi^\dagger_\sigma(x)\, \underset{\substack{\uparrow \\ \text{Pauli matrices}}}{\sigma^a_{\sigma\sigma'}}\, \psi_{\sigma'}(x) \longrightarrow$ spin density along $a = x, y, z$

Note: in principle for the true spin density we need a factor 1/2 but we leave the definition as that here).

- Let's have a closer look to $O_\rho(x)$. As for our discussion (p. 28) for the spin-less case we may split $O_\rho(x)$ into a $q \sim 0$ part and a $q \sim \pm 2k_F$ part:

$$O_\rho(x) = \psi^\dagger_\uparrow(x)\psi_\uparrow(x) + \psi^\dagger_\downarrow(x)\psi_\downarrow(x) = [\psi^\dagger_{R\uparrow} + \psi^\dagger_{L\uparrow}][\psi_{R\uparrow} + \psi_{L\uparrow}] + [\psi^\dagger_{R\downarrow} + \psi^\dagger_{L\downarrow}](\psi_{R\downarrow} + \psi_{L\downarrow}) =$$

$$= \underbrace{[\psi^\dagger_{R\uparrow}\psi_{R\uparrow} + \psi^\dagger_{L\uparrow}\psi_{L\uparrow}] + [\psi^\dagger_{R\downarrow}\psi_{R\downarrow} + \psi^\dagger_{L\downarrow}\psi_{L\downarrow}]}_{q\sim 0} + \underbrace{(\psi^\dagger_{R\uparrow}\psi_{L\uparrow} + \psi^\dagger_{L\uparrow}\psi_{R\uparrow}) + (\psi^\dagger_{R\downarrow}\psi_{L\downarrow} + \psi^\dagger_{L\downarrow}\psi_{R\downarrow})}_{q\sim \pm 2k_F}$$

$$\underbrace{\rho_{R\uparrow} + \rho_{L\uparrow} + \rho_{R\downarrow} + \rho_{L\downarrow}}_{-\frac{1}{\pi}(\nabla\phi_\uparrow + \nabla\phi_\downarrow) = -\frac{\sqrt{2}}{\pi}\nabla\phi_\rho}$$

p. 35

$$\longrightarrow \frac{1}{2\pi\alpha}\Big\{ e^{-2ik_Fx} e^{i\sqrt{2}(\phi_\rho+\phi_\sigma)} + e^{2ik_Fx} e^{-i\sqrt{2}(\phi_\rho+\phi_\sigma)} + e^{-2ik_Fx} e^{i\sqrt{2}(\phi_\rho-\phi_\sigma)} + e^{2ik_Fx} e^{-i\sqrt{2}(\phi_\rho-\phi_\sigma)} \Big\}$$

$$= \frac{1}{\pi\alpha}\Big[e^{i\sqrt{2}\phi_\rho} e^{-2ik_Fx} \cos(\sqrt{2}\phi_\sigma) + e^{-i\sqrt{2}\phi_\rho} e^{2ik_Fx} \cos(\sqrt{2}\phi_\rho) \Big]$$

- We introduce at this point the charge-density wave (CDW) operator:

$$\boxed{O_{CDW}(x) = \frac{e^{-2ik_Fx}}{\pi\alpha}\, e^{i\sqrt{2}\phi_\rho} \cos(\sqrt{2}\phi_\sigma)}$$

→ measures the $2k_F$ part of the charge density $\psi^\dagger_{R\uparrow}\psi_{L\uparrow} + \psi^\dagger_{R\downarrow}\psi_{L\downarrow}$

Hence: $\boxed{O_\rho(x) = -\frac{\sqrt{2}}{\pi}\nabla\phi_\rho + (O_{CDW}(x) + h.c.)}$

Note: In terms of the fermionic operators: $O_{CDW} = \psi^\dagger_{R\uparrow}\psi_{L\uparrow} + \psi^\dagger_{R\downarrow}\psi_{L\downarrow}$)

* Let's evaluate the density-density correlations.

As in p. 28 we will have a $1/x^2$ dependence coming from the $q \sim 0$ term ($\sim \langle \nabla\phi_\rho(\vec{r}) \nabla\phi_\rho(0)\rangle$ and along the calculation of p. 28), and then we will have the contribution coming from the $2k_F$ terms (i.e. from O_{CDW} now):

$$\langle O^\dagger_{CDW}(\vec{r})\, O_{CDW}(0)\rangle = \frac{e^{2ik_F x}}{(\pi\alpha)^2} \langle e^{-i\sqrt{2}[\phi_\rho(\vec{r})-\phi_\rho(0)]} \cos(\sqrt{2}\,\phi_\sigma(\vec{r}))\cos(\sqrt{2}\phi_\sigma(0))\rangle$$

here we employ the spin-charge separation, and that correlations with $\sum_j A_j \neq 0$ are zero (p. 25)

$$= \frac{e^{2ik_F x}}{4\pi^2\alpha^2} \langle e^{-i\sqrt{2}(\phi_\rho(\vec{r})-\phi_\rho(0))}\rangle \left[\langle e^{-i\sqrt{2}(\phi_\sigma(\vec{r})-\phi_\sigma(0))}\rangle + \langle e^{i\sqrt{2}(\phi_\sigma(\vec{r})-\phi_\sigma(0))}\rangle\right]$$

$$\overset{p.27}{=} \frac{e^{2ik_F x}}{2(\pi\alpha)^2} e^{-\langle(\phi_\rho(\vec{r})-\phi_\rho(0))^2\rangle} e^{-\langle(\phi_\sigma(\vec{r})-\phi_\sigma(0))^2\rangle} \overset{p. 23 \text{ and } 25}{=}$$

Here we take $g_{1\perp}=0$

$$= \frac{e^{2ik_F x}}{2(\pi\alpha)^2} e^{-\frac{K_\rho}{2}\ln\left(\frac{r^2}{\alpha^2}\right)} e^{-\frac{K_\sigma}{2}\ln\left(\frac{r^2}{\alpha^2}\right)}$$

Hence
$$\boxed{\langle O^\dagger_{CDW}(\vec{r})\, O_{CDW}(0)\rangle = \frac{e^{2ik_F x}}{2(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{K_\rho+K_\sigma}}$$

. As we may have expected we get that the $2k_F$ correlations show a non-universal power-law decay with an exponent that depends on interactions.

* Let's have a closer look now to $O^z_\sigma(x)$

$$O^z_\sigma(x) = (\psi^\dagger_{R\uparrow}\psi_{R\uparrow} + \psi^\dagger_{L\uparrow}\psi_{L\uparrow}) - (\psi^\dagger_{R\downarrow}\psi_{R\downarrow} + \psi^\dagger_{L\downarrow}\psi_{L\downarrow}) \quad \leftarrow q\sim 0 \text{ part}$$

$$+ (\psi^\dagger_{R\uparrow}\psi_{L\uparrow} + \psi^\dagger_{L\uparrow}\psi_{R\uparrow}) - (\psi^\dagger_{R\downarrow}\psi_{L\downarrow} + \psi^\dagger_{L\downarrow}\psi_{R\downarrow}) \quad \leftarrow q\sim 2k_F \text{ part.}$$

· Let's have a look to the $q \sim 0$ part

$$(\psi^+_{R\uparrow}\psi_{R\uparrow}+\psi^+_{L\uparrow}\psi_{L\uparrow})-(\psi^+_{R\downarrow}\psi_{R\downarrow}+\psi^+_{L\downarrow}\psi_{L\downarrow}) = (\rho_{R\uparrow}+\rho_{L\uparrow})-(\rho_{R\downarrow}+\rho_{R\downarrow}) = \frac{-1}{\pi}\nabla(\phi_\uparrow-\phi_\downarrow)$$
$$= -\frac{\sqrt{2}}{\pi}\nabla\phi_\sigma$$

· Let's see now the $q \sim 2k_F$ part

$$(\psi^+_{R\uparrow}\psi_{L\uparrow}+\psi^+_{L\uparrow}\psi_{R\uparrow}) - (\psi^+_{R\downarrow}\psi_{L\downarrow}+\psi^+_{L\downarrow}\psi_{R\downarrow}) =$$
$$= \frac{1}{2\pi\alpha}\left(e^{-i2k_Fx}e^{i\sqrt{2}(\phi_\rho+\phi_\sigma)}+h.c.\right)-\frac{1}{2\pi\alpha}\left(e^{-2ik_Fx}e^{i\sqrt{2}(\phi_\rho-\phi_\sigma)}+h.c\right)$$
$$= \frac{1}{2\pi\alpha}e^{-i2k_Fx}e^{i\sqrt{2}\phi_\rho}\left[e^{+i\sqrt{2}\phi_\sigma}-e^{-i\sqrt{2}\phi_\sigma}\right]+h.c.$$
$$= \frac{i}{\pi\alpha}e^{-i2k_Fx}e^{i\sqrt{2}\phi_\rho}\sin[\sqrt{2}\phi_\sigma]+h.c.$$

· We introduce at this point the spin-density-wave operator (SDW) along z:

$$\boxed{O^z_{SDW}(x) = \frac{i}{\pi\alpha}e^{-2ik_Fx}e^{i\sqrt{2}\phi_\rho}\sin[\sqrt{2}\phi_\sigma]}$$

measures the $2k_F$ part of the spin(S_z)-density

$$O^z_{SDW} = \psi^+_{R\uparrow}\psi_{L\uparrow}-\psi^+_{R\downarrow}\psi_{L\downarrow}$$

· Hence:

$$\boxed{O^z_\sigma(x) = -\frac{\sqrt{2}}{\pi}\nabla\phi_\sigma + (O^z_{SDW}(x)+h.c.)}$$

· Let's evaluate the correlation functions. Clearly the $q\sim 0$ part contributes again with $1/x^2$ (We consider here again $g_{1\perp}=0$ for simplicity of the current discussion). Let's see the part coming from the $2k_F$ terms:

$$\langle O^{z+}_{SDW}(\bar{r})\, O^z_{SDW}(0)\rangle = \frac{1}{(\pi\alpha)^2}e^{2ik_Fx}\langle e^{-i\sqrt{2}(\phi_\rho(\bar{r})-\phi_\rho(0))}\rangle\langle\sin\sqrt{2}\phi_\sigma(\bar{r})\sin\sqrt{2}\phi_\sigma(0)\rangle$$
$$= \frac{1}{(\pi\alpha)^2}e^{2ik_Fx}\langle e^{-i\sqrt{2}[\phi_\rho(\bar{r})-\phi_\rho(0)]}\rangle\frac{1}{2}\langle e^{i\sqrt{2}(\phi_\sigma(\bar{r})-\phi_\sigma(0))}\rangle$$
$$= \frac{1}{2\pi\alpha^2}e^{2ik_Fx}e^{-\langle(\phi_\rho(\bar{r})-\phi_\rho(0))^2\rangle}e^{-\langle(\phi_\sigma(\bar{r})-\phi_\sigma(0))^2\rangle}$$
$$= \frac{1}{2\pi\alpha^2}e^{2ik_Fx}e^{-\frac{K_\rho}{2}\ln\left(\frac{r^2}{\alpha^2}\right)}e^{-\frac{K_\sigma}{2}\ln\left(\frac{r^2}{\alpha^2}\right)}$$

· Hence:

$$\langle O^{z\,+}_{SDW}(\vec{r})\, O^{z}_{SDW}(0)\rangle = \frac{e^{2ik_F x}}{2(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{K_\rho+K_\sigma}$$

* Let's see now O^x_σ:

$$O^x_\sigma = \psi^\dagger_\uparrow \psi_\downarrow + \psi^\dagger_\downarrow \psi_\uparrow = (\psi^\dagger_{R\uparrow}+\psi^\dagger_{L\uparrow})(\psi_{R\downarrow}+\psi_{L\downarrow}) + (\psi^\dagger_{R\downarrow}+\psi^\dagger_{L\downarrow})(\psi_{R\uparrow}+\psi_{L\uparrow})$$

$$= (\psi^\dagger_{R\uparrow}\psi_{R\downarrow}+\psi^\dagger_{L\uparrow}\psi_{L\downarrow}) + (\psi^\dagger_{R\downarrow}\psi_{R\uparrow}+\psi^\dagger_{L\downarrow}\psi_{L\uparrow}) \longleftarrow q\sim 0 \text{ term}$$

$$+ (\psi^\dagger_{R\uparrow}\psi_{L\downarrow}+\psi^\dagger_{L\uparrow}\psi_{R\downarrow}) + (\psi^\dagger_{R\downarrow}\psi_{L\uparrow}+\psi^\dagger_{L\downarrow}\psi_{R\uparrow})$$

· Let's see the $q\sim 0$ term:

$$\frac{1}{2\pi\alpha}\left[e^{\frac{i}{\sqrt{2}}(\phi_\rho-\theta_\rho+\phi_\sigma-\theta_\sigma)}\, e^{i/\sqrt{2}(-\phi_\rho+\theta_\rho+\phi_\sigma-\theta_\sigma)} + h.c.\right]$$

$$+\frac{1}{2\pi\alpha}\left[e^{\frac{i}{\sqrt{2}}(-\phi_\rho-\theta_\rho-\phi_\sigma-\theta_\sigma)}\, e^{i/\sqrt{2}(\phi_\rho+\theta_\rho-\phi_\sigma-\theta_\sigma)} + h.c\right]$$

$$= \frac{2}{\pi\alpha}\cos(\sqrt{2}\,\phi_\sigma(x))\cos(\sqrt{2}\,\theta_\sigma(x))$$

* Let's see now the $q\sim 2k_F$ term

$$\psi^\dagger_{R\uparrow}\psi_{L\downarrow} + \psi^\dagger_{R\downarrow}\psi_{L\uparrow} = \frac{1}{2\pi\alpha}e^{-2ik_F x}\left\{ \begin{array}{l} e^{i/\sqrt{2}(\phi_\rho-\theta_\rho+\phi_\sigma-\theta_\sigma)}\, e^{-i/\sqrt{2}(-\phi_\rho-\theta_\rho+\phi_\sigma+\theta_\sigma)} \\ + e^{i/\sqrt{2}(\phi_\rho-\theta_\rho-\phi_\sigma+\theta_\sigma)}\, e^{-i/\sqrt{2}(-\phi_\rho-\theta_\rho-\phi_\sigma-\theta_\sigma)} \end{array}\right\}$$

$$= \frac{e^{-2ik_F x}}{(2\pi\alpha)}\left[e^{i\sqrt{2}(\phi_\rho-\theta_\sigma)} + e^{i\sqrt{2}(\phi_\rho+\theta_\sigma)}\right] = \frac{e^{-2ik_F x}}{\pi\alpha}\, e^{i\sqrt{2}\phi_\rho}\cos(\sqrt{2}\,\theta_\sigma)$$

We can introduce here the spin-density-wave gender (along x):

$$\boxed{O^x_{SDW}(x) = \frac{e^{-2ik_F x}}{\pi\alpha}\, e^{i\sqrt{2}\phi_\rho}\cos(\sqrt{2}\,\theta_\sigma)} \Longrightarrow O^x_{SDW} = \psi^\dagger_{R\uparrow}\psi_{L\downarrow}+\psi^\dagger_{R\downarrow}\psi_{L\uparrow}$$

· Hence:

$$\boxed{O^x_\sigma(x) = \frac{2}{\pi\alpha}\cos(\sqrt{2}\,\phi_\sigma(x))\cos(\sqrt{2}\,\theta_\sigma(x)) + \left[O^x_{SDW}(x) + h.c.\right]}$$

· The correlation of the $q\sim 0$ is slightly more involved than for θ_ρ and θ^z_σ

$$\langle\left(\frac{2}{\pi\alpha}\right)^2\cos(\sqrt{2}\,\phi_\sigma(\vec{r}))\cos(\sqrt{2}\,\theta_\sigma(\vec{r}))\cos(\sqrt{2}\,\phi_\sigma(0))\cos(\sqrt{2}\,\theta_\sigma(0))\rangle =$$

$$= \frac{1}{2}\left(\frac{1}{\pi\alpha}\right)^2\left[\langle e^{-i\sqrt{2}\left[[\phi_\sigma(\vec{r})-\theta_\sigma(\vec{r})]-[\phi_\sigma(0)-\theta_\sigma(0)]\right]}\rangle + \langle e^{-i\sqrt{2}\left[[\phi_\sigma(\vec{r})+\theta_\sigma(\vec{r})]-[\phi_\sigma(0)+\theta_\sigma(0)]\right]}\rangle\right.$$

$$= \frac{1}{2\pi^2\alpha^2}\, e^{-(K_\sigma+K_\sigma^{-1})F_1(\vec{r})}\left[1+e^{-2F_2(\vec{r})}\right] = \frac{1}{(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{\frac{K_\sigma+K_\sigma^{-1}}{2}}\left(\frac{iy_\alpha}{x+iy_\alpha}\right)$$

* Note that the $q\sim 0$ term goes as $1/r^2$ only if $K_\sigma = 1$.

- For the $q\sim 2k_F$ part:

$$\langle O^{x\,+}_{SDW}(\vec{r})\, O^{x}_{SDW}(0)\rangle = \frac{e^{2ik_F x}}{(\pi\alpha)^2} \langle e^{-i\sqrt{2}(\phi_\rho(\vec{r})-\phi_\rho(0))}\rangle \langle \cos\sqrt{2}\theta_\sigma(\vec{r}) \cos\sqrt{2}\theta_\sigma(0)\rangle$$

$$= \frac{e^{2ik_F x}}{2(\pi\alpha)^2} \langle e^{-i\sqrt{2}(\phi_\rho(\vec{r})-\phi_\rho(0))}\rangle \langle e^{-i\sqrt{2}(\theta_\sigma(\vec{r})-\theta_\sigma(0))}\rangle$$

$$= \frac{e^{2ik_F x}}{2(\pi\alpha)^2} e^{-\frac{K_\rho}{2}\ln\left(\frac{r^2}{\alpha^2}\right)} e^{-\frac{1}{2K_\sigma}\ln\left(\frac{r^2}{\alpha^2}\right)}$$

Hence
$$\boxed{\langle O^{x\,+}_{SDW}(\vec{r})\, O^{x}_{SDW}(0)\rangle = \frac{e^{2ik_F x}}{2\pi\alpha} \left(\frac{\alpha}{r}\right)^{K_\rho + 1/K_\sigma}}$$

- Finally:
$$\boxed{O^{y}_{\sigma}(x) = -\frac{2}{\pi\alpha} \cos(\sqrt{2}\phi_\sigma^{(x)}) \sin(\sqrt{2}\theta_\sigma^{(x)}) + [O^{y}_{SDW}(x) + h.c.]}$$

with
$$\boxed{O^{y}_{SDW}(x) = -i(\psi^\dagger_{R\uparrow}\psi_{L\downarrow} + \psi^\dagger_{R\downarrow}\psi_{L\uparrow}) = -\frac{e^{-2ik_F x}}{\pi\alpha} e^{i\sqrt{2}\phi_\rho} \sin(\sqrt{2}\theta_\sigma)}$$

$O^{y}_{SDW} = -i[\psi^\dagger_{R\uparrow}\psi_{L\downarrow} - \psi^\dagger_{R\downarrow}\psi_{L\uparrow}]$

- The $q\sim 0$ part has a correlation $\frac{1}{(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{\frac{K_\sigma + K_\sigma^{-1}}{2}} \left(\frac{-i\,y_\alpha}{x - i\,y_\alpha}\right)$

whereas
$$\boxed{\langle O^{y}_{SDW}(\vec{r})\, O^{y}_{SDW}(0)\rangle = \frac{e^{2ik_F x}}{2(\pi\alpha)^2} \left(\frac{\alpha}{r}\right)^{K_\rho + 1/K_\sigma}}$$

- It's easy to see that for non-interacting gases ($K_\rho = K_\sigma = 1$) all correlations decay as $1/r^2$ as expected for a free-fermion case. In that case $O^{x,y,z}_{SDW}$ have the same decay as imposed by spin rotation symmetry. This rotational symmetry is preserved in the interacting case if $K_\sigma = 1$. Note that if the interactions obey spin rotation symmetry (i.e. they are spin-independent), then $g_\parallel = g_\perp$ and from p. 37 then $K_\sigma = 1$ (if $g_{\parallel} = 0$). If $g_\parallel \neq g_\perp \longrightarrow K_\sigma \neq 1$ and the spin-rotation symmetry is broken.

- Pairing operators: singlet and triplet pairing

* As for the spin-less case (p. 29) we may define the pairing operators:

  - Singlet pairing: $O_{SS}(x) = \sum_{\sigma,\sigma'} \sigma\, \psi^+_{R\sigma}(x)\, \delta_{\sigma\sigma'}\, \psi^+_{L,-\sigma'}(x)$

  - Triplet pairing ($a = x, y, z$): $O^a_{TS}(x) = \sum_{\sigma,\sigma'} \sigma\, \psi^+_{R\sigma}(x)\, \sigma^a_{\sigma\sigma'}\, \psi^+_{L,-\sigma'}(x)$

Note: The names singlet and triplet have a clear justification:
$O_{SS} \equiv \psi^+_{R\uparrow}\psi^+_{L\downarrow} - \psi^+_{R\downarrow}\psi^+_{L\uparrow}$ ($\equiv |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$), $O^z_{TS} \equiv \psi^+_{R\uparrow}\psi^+_{L\downarrow} + \psi^+_{R\downarrow}\psi^+_{L\uparrow}$ ($\equiv |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$)
$O^x_{TS} \equiv \psi^+_{R\uparrow}\psi^+_{L\uparrow} - \psi^+_{R\downarrow}\psi^+_{L\downarrow}$ ($\equiv |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle$), $O^y_{TS} \equiv -i(\psi^+_{R\uparrow}\psi^+_{L\uparrow} + \psi^+_{R\downarrow}\psi^+_{L\downarrow})$ ($\equiv |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$)

Note II: For spin-less fermions (p. 29) pairing with momentum non zero (e.g. $\psi^+_R \psi^+_R$) was largely suppressed due to Pauli-blocking. Here it isn't the case since we have 2 different spins. The singlet and triplet pairing describe pairing with zero total momentum)

* We may express the singlet- and triplet pairing as a function of $\phi_{\rho,\sigma}$ and $\theta_{\rho,\sigma}$:

$$O^z_{TS}(x) = \frac{1}{\pi\alpha} e^{-i\sqrt{2}\theta_\rho} \sin\sqrt{2}\phi_\sigma$$

$$O^x_{TS}(x) = \frac{1}{\pi\alpha} e^{-i\sqrt{2}\theta_\rho} \cos\sqrt{2}\theta_\sigma$$

$$O^y_{TS}(x) = \frac{-1}{\pi\alpha} e^{-i\sqrt{2}\theta_\rho} \sin\sqrt{2}\theta_\sigma$$

$$O_{SS}(x) = \frac{1}{\pi\alpha} e^{-i\sqrt{2}\theta_\rho} \cos(\sqrt{2}\phi_\sigma)$$

(Note: there's some kind of arbitrariness since we employ e.g. $O_{SS} = \psi^+_{R\uparrow}\psi^+_{L\downarrow} + \psi^+_{L\uparrow}\psi^+_{R\downarrow}$ to get the expression of O^z_{TS} besides. If we anticommute then $O_{SS} \to \psi^+_{R\uparrow}\psi^+_{L\downarrow} - \psi^+_{R\downarrow}\psi^+_{L\uparrow}$ and we'll get a ~~cosine. There's a way to sort out this~~ to get that the written result is the correct one but we won't see this by now.) (see p. 114)

- It's clear that we will have again power-law decays

- Up to now we have considered $g_\perp = 0$. In the following we will employ renormalization group techniques to study the effects of the $g_\perp$ terms.

* Renormalization-group equations for the sine-Gordon Hamiltonian

* Wilson-like approach.

* We will see now how to treat the sine-Gordon Hamiltonian using renormalization group (RG) techniques. We will first follow a Wilson-like approach.

* Recall that the Hamiltonian of the sine-Gordon is of the form: (p. 38)

$$H = \frac{1}{2\pi}\int dx \left[uK(\nabla\theta)^2 + \frac{u}{K}(\nabla\phi)^2 \right] + \frac{2g}{(2\pi\alpha)^2}\int dx \cos(2\sqrt{2}\phi)$$

(Note: Here I remove the subindex "σ", and $g \equiv g_{1\perp}$)

* We will need in the following the action in imaginary time (τ)

$$S = \int_0^\beta d\tau \int dx \left\{ -\frac{i}{\pi}\nabla\theta\, \partial_\tau\phi + \mathcal{H} \right\}$$

Note: recall that the action is typically defined (in real time) as

$$S = \int dt \int dx\, \mathcal{L}(x,t) = \int dt \int dx \left[\Pi(x,t)\,\partial_t\phi(x,t) - \mathcal{H}(x,t) \right] \stackrel{t=-i\tau}{=} \quad \Pi(x,t) = \frac{\nabla\theta(x,t)}{\pi} \text{ p. 13}$$

$$= -i\int d\tau \int dx \left[\frac{1}{\pi}\nabla\theta(x,\tau)\, i\partial_\tau\phi - \mathcal{H} \right] = iS \leftarrow \text{action in imaginary time})$$

* The Heisenberg equation (in imaginary time) for ϕ is: p. 13

$$-\partial_\tau\phi(x) = [H, \phi(x)] = \frac{1}{2\pi}\int dx'\, uK \left[[\nabla\theta(x')]^2, \phi(x) \right] \stackrel{!}{=} \frac{1}{2\pi} 2uK\,(i\pi)\nabla\theta(x) = iuK\nabla\theta(x)$$

Hence $\boxed{\nabla\theta(x) = \frac{i}{uK}\partial_\tau\phi(x)}$

Thus: $-\frac{i}{\pi}\nabla\theta\,\partial_\tau\phi + \frac{1}{2\pi}uK(\nabla\theta)^2 = \frac{1}{\pi uK}(\partial_\tau\phi)^2 - \frac{uK}{2\pi}\frac{1}{(uK)^2}(\partial_\tau\phi)^2 = \frac{(\partial_\tau\phi)^2}{2\pi uK}$

* And then, the action S can be written in the form:

$$S = \int_0^\beta d\tau \int dx \left\{ \underbrace{\frac{(\partial_\tau\phi)^2}{2\pi uK} + \frac{u}{K}(\nabla\phi)^2}_{\text{harmonic part}} + \frac{2g}{(2\pi\alpha)^2}\cos(2\sqrt{2}\phi) \right\}$$

For simplicity we'll assume $\beta = \infty$ ($T = 0$).

* We Fourier-Transform the field $\phi(x,\tau)$:

$$\phi(x,\tau) = \frac{1}{\beta\Omega}\sum_{k,\omega_n} e^{i(kx-\omega_n\tau)}\phi(k,\omega_n)$$

* Let's impose a sharp momentum cut-off to start with $\Rightarrow \Lambda$

We vary the cut-off between Λ and Λ'.

We can decompose:

$$\phi(r) = \underbrace{\phi^{>}(r)}_{\text{fast modes}} + \underbrace{\phi^{<}(r)}_{\text{slow modes}}$$

$$\phi^{<}(\vec{r}) = \frac{1}{\beta\Omega}\sum_{|\vec{q}|<\Lambda'} e^{i\vec{q}\cdot\vec{r}}\phi(\vec{q})$$

$$\phi^{>}(\vec{r}) = \frac{1}{\beta\Omega}\sum_{\Lambda'<|q|<\Lambda} e^{i\vec{q}\cdot\vec{r}}\phi(\vec{q})$$

* Let's analyze now the quadratic part of the action:

$$S_0 = \frac{1}{2\pi K}\int_0^{\infty} d\tau \int_{-\infty}^{\infty} dx \left\{ \frac{1}{u}(\partial_\tau\phi)^2 + u(\nabla\phi)^2 \right\}$$

$$= \frac{1}{2\pi K}\frac{1}{(\beta\Omega)^2}\sum_{\vec{q},\vec{q}'}\left[\frac{(-i\omega_n)(-i\omega_{n'})}{u} + u(ik)(ik')\right]\underbrace{\left[\int e^{i(\vec{q}+\vec{q}')\vec{r}}\, d\tau dx\right]}_{\beta\Omega\,\delta_{\vec{q}',-\vec{q}}}\phi(\vec{q})\phi(\vec{q}')$$

$$\overset{\phi(\vec{q})^{*}=\phi(-\vec{q})}{=} \frac{1}{2\pi K}\frac{1}{\beta\Omega}\sum_{\vec{q}}\left[\frac{\omega_n^2}{u} + uk^2\right]\phi(\vec{q})^{*}\phi(\vec{q})$$

* Clearly we can write $\longrightarrow S_0 = S_0^{>} + S_0^{<}$

Then the corresponding partition function for S_0 is

$$Z_0 = \int D\phi\, e^{-S_0} = Z_0^{>}\cdot Z_0^{<}$$

* Then: $dx\,d\tau = \frac{1}{u}d^2r$

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int D\phi\, e^{-S_0}\, e^{-\int d^2r \frac{2g}{(2\pi\alpha)^2 u}\cos\sqrt{8}\phi} \overset{\text{expanding up to second order in powers of the cosine term}}{=}$$

$$= \frac{1}{Z_0}\int D\phi\, e^{-S_0}\left\{1 - \int d^2r \frac{2g}{(2\pi\alpha)^2 u}\cos\sqrt{8}\phi + \frac{1}{2}\left(\frac{2g/u}{(2\pi\alpha)^2}\right)^2\int d^2r_1\int d^2r_2 \cos(\sqrt{8}\phi(r_1))\cos(\sqrt{8}\phi(r_2))\right.$$

- We split now into > and < parts:

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int D\phi\, e^{-S_0^>} e^{-S_0^<}\Big\{1 - \int d^2r \frac{2g}{(2\pi\alpha)^2 u}\cos[\sqrt{8}(\phi^>(r)+\phi^<(r))]$$
$$+ \frac{2g^2}{u^2(2\pi\alpha)^4}\int d^2r_1 \int d^2r_2 \cos\left[\sqrt{8}(\phi^>(r_1)+\phi^<(r_1))\right]\cos\left[\sqrt{8}[\phi^>(r_2)+\phi^<(r_2)]\right]\Big\}$$

- We will now do averages over the > parts, this means:

$$\langle O\rangle_> = \frac{1}{Z_0^>}\int D\phi\, e^{-S_0^>} O$$

- Let's see the term:

$$\langle\cos[\sqrt{8}(\phi^> + \phi^<)]\rangle_> = \cos(\sqrt{8}\phi^<)\ \langle\cos\sqrt{8}\phi^>\rangle_> - \sin(\sqrt{8}\phi^<)\langle\sin\sqrt{8}\phi^>\rangle_>$$

Note that:

$$\langle\cos\sqrt{8}\phi^>\rangle_> = \frac{1}{2}\left[\langle e^{i\sqrt{8}\phi^>}\rangle_> + \langle e^{-i\sqrt{8}\phi^>}\rangle_>\right] = \frac{1}{2}\left[e^{-4\langle\phi^>(r)^2\rangle_>} + e^{-4\langle\phi^>(r)^2\rangle_>}\right]$$
$$= e^{-4\langle\phi^>(r)^2\rangle}$$

$$\langle\sin\sqrt{8}\phi^>\rangle_> = 0$$

Hence:

$$\langle\cos[\sqrt{8}(\phi^> + \phi^<)]\rangle_> = e^{-4\langle\phi^>(r)^2\rangle}\ \cos(\sqrt{8}\phi^<(r))$$

- Similarly:

$$\langle\cos[\sqrt{8}(\phi_1^> + \phi_1^<)]\cos(\sqrt{8}(\phi_2^> + \phi_2^<))\rangle_> =$$
$$= \cos\sqrt{8}\phi_1^<\ \cos\sqrt{8}\phi_2^<\ \langle\cos\sqrt{8}\phi_1^>\cos\sqrt{8}\phi_2^>\rangle_> + \sin\sqrt{8}\phi_1^<\sin\sqrt{8}\phi_2^<\ \langle\sin\sqrt{8}\phi_1^>\sin\sqrt{8}\phi_1^>\rangle_>$$
$$- \cos\sqrt{8}\phi_1^<\sin\sqrt{8}\phi_2^<\langle\cos\sqrt{8}\phi_1^>\sin\sqrt{8}\phi_2^>\rangle_> - \sin\sqrt{8}\phi_1^<\cos\sqrt{8}\phi_2^<\ \langle\sin\sqrt{8}\phi_1^>\cos\sqrt{8}\phi_2^>\rangle_>$$

It's easy to see that

$$\langle\cos\sqrt{8}\phi_1^>\cos\sqrt{8}\phi_2^>\rangle_> = \frac{1}{2}\left\{e^{-4\langle(\phi_1^> - \phi_2^>)^2\rangle_>} + e^{-4\langle(\phi_1^> + \phi_2^>)\rangle_>}\right\}$$
$$\langle\sin\sqrt{8}\phi_1^>\sin\sqrt{8}\phi_2^>\rangle_> = \frac{+1}{2}\left\{e^{-4\langle(\phi_1^> - \phi_2^>)^2\rangle_>} - e^{-4\langle(\phi_1^> + \phi_2^>)\rangle_>}\right\}$$

$$\langle\cos\sqrt{8}\phi_1^>\sin\sqrt{8}\phi_1^>\rangle_> = 0$$

Hence:

$$\langle \cos[\sqrt{8}(\phi_1^> + \phi_1^<)] \cos[\sqrt{8}(\phi_2^> + \phi_2^<)] \rangle_> = \frac{1}{2} e^{-4\langle(\phi_1^> - \phi_2^>)^2\rangle} \cos\sqrt{8}(\phi_1^< - \phi_2^<) + \frac{1}{2} e^{-4\langle(\phi_1^> + \phi_2^>)^2\rangle} \cos\sqrt{8}(\phi_1^< + \phi_2^<)$$

Hence:

$$\frac{Z}{Z_0} = \frac{1}{Z_0^<} \int D\phi\, e^{-S_0^<} \left\{ 1 - \frac{2g}{(2\pi\alpha)^2 u} \int d^2r \cos(\sqrt{8}\phi^<(r))\, e^{-4\langle \phi^>(r)^2\rangle_>} + \frac{g^2}{(2\pi\alpha)^4 u^2} \sum_{\epsilon=\pm 1} \int d^2r_1 \int d^2r_2 \cos\sqrt{8}[\phi^<(r_1) + \epsilon\phi^<(r_2)]\, e^{-4\langle[\phi^>(r_1) + \epsilon\phi^>(r_2)]^2\rangle} \right\}$$

- We will now re-exponenciate. Note that (up to second order in g) we may re-write:

$$\frac{Z}{Z_0} = \frac{1}{Z_0^<} \int D\phi\, e^{-S_0^<}\, e^{-\frac{2g}{(2\pi\alpha)^2 u} \int d^2r \cos(\sqrt{8}\phi(r))\, e^{-4\langle\phi^>(r)^2\rangle_>}}$$
$$\times\, e^{\frac{g^2}{(2\pi\alpha)^4 u^2} \sum_{\varepsilon=\pm 1} \int d^2r_1 \int d^2r_2 \cos\sqrt{8}(\phi_1^< + \varepsilon\phi_2^<)\, e^{-4\langle(\phi_1^> + \varepsilon\phi_2^>)^2\rangle_>}}$$
$$\times\, e^{\frac{-2g^2}{(2\pi\alpha)^4 u^2} \int d^2r_1 \int d^2r_2 \cos(\sqrt{8}\phi_1^>) \cos(\sqrt{8}\phi_2^>)\, e^{-4\langle\phi_1^{>2}\rangle_>}\, e^{-4\langle\phi_2^{>2}\rangle_>}}$$

(Note: the last term appears to compensate the second-order term of the first exponential)

- The first exponential is like the original cosine but for the slow fields only. Let's first have a look to this term.

To get back to an action identical to the original one we must re-scale distance and time to bring back the cut-off to its original value. This can be done by ~~code~~ defining:

$$\begin{cases} dk = \frac{\Lambda'}{\Lambda} dk' \\ d\omega = \frac{\Lambda'}{\Lambda} d\omega \end{cases} \qquad \begin{cases} dx = \frac{\Lambda}{\Lambda'} dx' \\ d\tau = \frac{\Lambda}{\Lambda'} d\tau' \end{cases}$$

Hence: $$-\frac{2g}{(2\pi\alpha)^2 u} \int d^2r \cos(\sqrt{8}\phi^<)\, e^{-4\langle\phi^{>2}\rangle_>} = \frac{-2}{(2\pi\alpha)^2 u} \int d^2r' \cos(\sqrt{8}\phi^<) \left(\frac{\Lambda}{\Lambda'}\right)^2 g\, e^{-4\langle\phi^{>2}\rangle_>}$$

We have then a new coupling constant

$$g(\Lambda') = \left(\frac{\Lambda}{\Lambda'}\right)^2 g(\Lambda)\, e^{-4\langle\phi^>(r)^2\rangle_>}$$

Note that

$$-4\langle\phi^2(\vec r)\rangle_> = -4\frac{1}{(\beta\Omega)^2}\sum_{q,q'} e^{i(\vec q+\vec q')\cdot\vec r}\langle\phi(q)\phi(q')\rangle_> \overset{P(23)}{=}$$

$$= -4\frac{1}{\beta\Omega}\pi K\sum_q^{>}\frac{1}{\omega_n^2/u+uk^2} \longrightarrow -4\pi K\int_>\frac{dk\,d\omega}{(2\pi)^2}\frac{u}{\omega^2+u^2k^2} =$$

$$= -4\pi K\int_>\frac{dk\,d\left(\frac{\omega}{u}\right)}{(2\pi)^2}\frac{1}{\left(\frac{\omega}{u}\right)^2+k^2} = -\frac{4\pi K}{4\pi^2}\int_{\Lambda'}^{\Lambda}\frac{2\pi q\,dq}{q^2} = -2K\ln\left(\frac{\Lambda}{\Lambda'}\right)$$

Hence:

$$g(\Lambda') = g(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^2 e^{-2K\ln\left(\frac{\Lambda}{\Lambda'}\right)}$$

- If we parametrize (as usual): $\Lambda(l)=\Lambda_0 e^{-l}$ where Λ_0 is the bare cut-off, and one makes the infinitesimal change $\Lambda'=\Lambda_0 e^{-l-dl}$ one gets: $g(l+dl) = g(l)e^{2\delta l}e^{-2K\ln(e^{\delta l})} = g(l)e^{(2-2K)dl}$

Hence: $\boxed{\frac{dg(l)}{dl} = (2-2K)g(l)}$

- This equation is already one of the RG equations. We must now have a look to the rest of the terms in the action in p. 49.

We can combine the exponents of these terms into:

$$\frac{g^2}{(2\pi\alpha)^4u^2}\int d^2r_1\int d^2r_2\left\{\cos\left(\sqrt8(\phi_1^<-\phi_2^<)\right)\left[e^{-4\langle(\phi_1^>-\phi_2^>)^2\rangle}-e^{-4[\langle\phi_1^{>2}\rangle+\langle\phi_2^{>2}\rangle]}\right]\right.$$
$$\left.+\cos\left(\sqrt8(\phi_1^<+\phi_2^<)\right)\left[e^{-4\langle(\phi_1^>+\phi_2^>)^2\rangle}-e^{-4[\langle\phi_1^{>2}\rangle+\langle\phi_2^{>2}\rangle]}\right]\right\}$$

clearly

- One may clearly see that the main contribution comes for $\langle(\phi_1^>\pm\phi_2^>)^2\rangle$ clearly different than $\langle\phi_1^{>2}\rangle+\langle\phi_2^{>2}\rangle$, i.e. when $\langle\phi_1^>\phi_2^>\rangle$ is clearly different than zero.

One may see that

$$\langle\phi_1^>\phi_2^>\rangle_> = \frac{1}{\beta\Omega}\sum_q^{>}\frac{\pi K}{\frac{\omega_n^2}{u}+uk^2}e^{i\vec q(\vec r_1-\vec r_2)} \longrightarrow \frac{\pi K}{(2\pi)^2}\int_> dk\int d\omega\frac{e^{i\vec q(x,u\tau)}}{\omega^2+k^2} = \frac{\pi K}{(2\pi)^2}\int_{\Lambda'}^{\Lambda}\frac{dq}{q}\int_0^{2\pi}e^{i\vec q\cdot\vec r_{12}}d\theta$$

$$= \frac{\pi K}{2\pi}\int_{\Lambda'}^{\Lambda}\frac{dq}{q}J_0(qr_{12})$$ clearly $\langle\phi_1^>\phi_2^>\rangle_>$ tends to zero when r_1 and r_2 are far away.

· The main contribution comes from the region where r_1 and r_2 are close. Hence the term with $\cos(\sqrt{8}(\phi_1+\phi_2))$ goes essentially as $\cos(2\sqrt{8}\phi(r))$ where $r_1 \sim r_2 \sim r$. This term is a new cosine but instead of $\cos(\sqrt{8}\phi)$ we have $\cos(\sqrt{8}\, 2\phi)$. It's easy to see that this may be renormalized as the previous term, but now it has $8K$ instead of $2K$ (i.e. $g \sim e^{-8Kdl}$ instead of e^{2Kdl}) and hence it decays fast.

· We will hence retain only the contribution with $\epsilon = -$:

recall p.23

$$\frac{g^2}{(2\pi\alpha)^4 u^2}\int d^2r_1 \int d^2 r_2 \cos[\sqrt{8}(\phi_1^< - \phi_2^<)]\left\{e^{-4\langle(\phi_1^< - \phi_2^<)^2\rangle} - e^{-4[\langle\phi_1^{<2}\rangle + \langle\phi_2^{<2}\rangle]}\right\}$$

$\vec{r} = \vec{r}_1 - \vec{r}_2$

$$= \frac{g^2}{(2\pi\alpha)^4 u^2}\int d^2r_1 \int d^2 r_2 \cos[\sqrt{8}(\phi_1^< - \phi_2^<)]\left[e^{-4\sum_{\Lambda'<q<\Lambda}[2-2\cos\vec{q}\vec{r}]\frac{\pi K u}{\omega_n^2 + K^2u^2}} - e^{-4\sum_{\Lambda'<q<\Lambda}\frac{2\pi K u}{\omega_n^2+K^2u^2}}\right]$$

$$= \frac{g^2}{(2\pi\alpha)^4 u^2}\int d^2r_1 \int d^2 r_2 \cos[\sqrt{8}(\phi_1^< - \phi_2^<)]\, e^{-\frac{4}{\beta\Omega}\sum_{\Lambda'<q<\Lambda}(2-2\cos\vec{q}\vec{r})\frac{\pi K u}{\omega_n^2+K^2u^2}} \times\left\{1 - e^{-\frac{4}{\beta\Omega}\sum_{\Lambda'<q<\Lambda} 2\cos(\vec{q}\cdot\vec{r})\frac{\pi K u}{\omega_n^2+u^2K^2}}\right\} = \delta I$$

· Note that the integral over q is just between Λ' and Λ with $\Lambda' \sim \Lambda$ hence only r values close to $1/\Lambda$ contribute (this is e.g. clear at the bottom of p. 50).

This term tends to zero when $\Lambda' \to \Lambda$ i.e. when $dl \to 0$, It goes ~~as~~ dl.

· We will now expand the term in bracket

$$1 - e^{-\frac{4}{\beta\Omega}\sum_{\Lambda'<q<\Lambda} 2\cos(\vec{q}\cdot\vec{r})\frac{\pi K u}{\omega_n^2+u^2K^2}} \simeq \frac{4}{\beta\Omega}\sum_{\Lambda'<q<\Lambda} 2\cos\vec{q}\cdot\vec{r}\,\frac{\pi K u}{\omega_n^2+u^2K^2}$$

$$\longrightarrow 4\int d\omega \int dk\, 2e^{i\vec{q}\cdot\vec{r}}\frac{\pi K u}{\omega^2+u^2k} \overset{p.50}{=} 4K\int_{\Lambda'}^{\Lambda}\frac{dq}{q}J_0(qr)$$

$$\simeq 4K J_0(\Lambda r)\ln(\Lambda/\Lambda') = 4K J_0(\Lambda r)\, dl$$

· We make also an expansion of the cosine $\cos(\sqrt{8}(\phi_1^< - \phi_2^<))$ in powers of r, we introduce $R = (r_1+r_2)/2$. With the expansion of the cosine there's however a technical problem, since the field ϕ is unbound $\langle\phi^2\rangle = \infty$ → recall that $\langle\phi^2\rangle = \frac{K}{\beta\Omega}\sum_q \frac{2\pi u}{\omega^2+u^2k^2}$ which has an ultraviolet logarithmic divergence)

* The cosine can't be expand directly. To do it safely one needs to normal order the cosine:

$$\cos\phi = :\cos\phi:\, e^{-\frac{1}{2}\langle\phi^2\rangle}$$

the normal ordered cosine can be expand safely!

$$\cos[\sqrt{8}(\phi_1^<-\phi_2^<)] \cong 4\,(\vec{r}\cdot\vec{\nabla}_R\phi(\vec{R}))\, e^{-\frac{4}{\beta\Omega}\sum_{q<\Lambda}(2-2\cos\vec{q}\cdot\vec{r})\frac{\pi K u}{\omega_n^2+u^2k^2}}$$

· Hence

$$\delta J = \frac{16 g^2 K dl}{(2\pi\alpha)^4 u^2}\int d^2R\int d^2r\,(\vec{r}\cdot\vec{\nabla}_R\phi(\vec{R}))^2\, e^{-4K F_{1\Lambda}(\vec{r})} J_0(\Lambda r)$$

$$\text{With } F_{1\Lambda}(\vec{r}) = \frac{1}{\beta\Omega}\sum_{q<\Lambda}[2-2\cos\vec{q}\cdot\vec{r}]\frac{\pi u}{\omega_n^2+u^2k^2} = \int_0^\Lambda \frac{dq}{q}(1-J_0(qr)) =$$

$$= F_1(\Lambda r) = \int_0^1\frac{dq}{q}[1-J_0(q(\Lambda r))]$$

· Then:

$$\delta J = \frac{16 g^2 K dl}{(2\pi\alpha)^4 u^2}\int d^2R\int r\,dr\, r^2 e^{-4KF_1(\Lambda r)} J_0(\Lambda r)\int_0^{2\pi} d\theta \begin{bmatrix}\cos^2\theta\,(\partial_x\phi)^2\\ +\sin^2\theta(\partial_y\phi)^2\\ +2\cos\theta\sin\theta(\partial_x\phi)(\partial_y\phi)\end{bmatrix}$$

$$= \frac{16\pi g^2 K dl}{(2\pi\alpha)^4 u^2}\int d^2R\,((\partial_x\phi)^2+(\partial_y\phi)^2)\int dr\, r^3 e^{-4KF_1(\Lambda r)} J_0(\Lambda r)$$

$$= \frac{16\pi g^2 K dl}{(2\pi\alpha)^4\Lambda^4 u^2}\int d^2R\,[(\partial_x\phi)^2+(\partial_y\phi)^2]\int dr\, r^3 e^{-4KF_1(r)} J_0(r)$$

$$= \frac{1}{2\pi}\Bigg[\underbrace{\frac{32\pi^2 g^2 K dl}{(2\pi\alpha)^4\Lambda^4 u^2}\int dr\, r^3 e^{-4KF_1(r)} J_0(r)}_{C}\Bigg]\int d\tau\int dx\left[\frac{(\partial_\tau\phi)^2}{u}+u(\nabla\phi)^2\right]$$

This has exactly the same form as S_0 (p. 47). Adding δJ to S_0 we get a renormalization of K:

$$\frac{1}{K(l+dl)} = \frac{1}{K(l)} + \frac{2g^2K(l)}{u^2\pi^2(\Lambda\alpha)^4}C\,dl$$

Hence $\boxed{\dfrac{dK^{-1}(\ell)}{d\ell} = \dfrac{2g^2 K(\ell)}{(4\pi)^2(\Lambda\alpha)^4}C}$ ← This is the 2nd RG equation

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• RG Equations: Derivation based on correlation function

* let's see another way of obtaining the RG equations, this time based in real-space cut-offs (and not in momentum space as before).

- let's consider the correlation

$$R(r_1-r_2) = \langle e^{i a\sqrt{2}\,\phi(r_1)}\, e^{-ia\sqrt{2}\,\phi(r_2)}\rangle_H$$

where $H = H_0 + \dfrac{2g}{(2\pi\alpha)^2}\displaystyle\int dx \cos(\sqrt{8}\phi)$ is our sine-Gordon Hamiltonian.

* The partition function is

$$Z = \int D\phi\, e^{-S} = \int D\phi\, e^{-S_0}\, e^{-\frac{2g}{(2\pi\alpha)^2 u}\int d^2r' \cos(\sqrt{8}\phi(r'))} \underset{\substack{\text{expanding for}\\ \text{small } g}}{\cong}$$

$$\cong \int D\phi\, e^{-S_0}\left\{1 - \frac{2g}{(2\pi\alpha)^2 u}\int d^2r' \cos\sqrt{8}\phi(r') + \frac{1}{2}\left(\frac{2g}{(2\pi\alpha)^2 u}\right)^2 \int d^2r'\int d^2r'' \cos\sqrt{8}\phi(r')\cos\sqrt{8}\phi(r'')\right\}$$

$$= Z_0\left\{1 - \frac{2g}{(2\pi\alpha)^2 u}\int d^2r' \underbrace{\langle\cos\sqrt{8}\phi(r')\rangle_{H_0}}_{0} + \frac{1}{2}\left(\frac{2g}{(2\pi\alpha)^2 u}\right)^2\int d^2r'\int d^2r''\langle\cos\sqrt{8}\phi(r')\cos\sqrt{8}\phi(r'')\rangle_{H_0}\right\}$$

* Let $\hat{O} = e^{ia\sqrt{2}\phi(r_1)}\, e^{-ia\sqrt{2}\phi(r_2)}$, then:

$$\langle\hat{O}\rangle_H = \frac{1}{Z}\int D\phi\, \hat{O}\, e^{-S} \cong \frac{1}{Z}\int D\phi\, \hat{O}\, e^{-S_0}\left\{1 - \frac{2g}{(2\pi\alpha)^2 u}\int d^2r' \cos\sqrt{8}\phi(r') + \frac{1}{2}\left(\frac{2g}{(2\pi\alpha)^2 u}\right)^2\int d^2r'\int d^2r''\cos\sqrt{8}\phi(r')\cos\sqrt{8}\phi(r'')\right\}$$

$$= \frac{Z_0}{Z}\left\{\langle\hat{O}\rangle_{H_0} - \frac{2g}{(2\pi\alpha)^2 u}\int d^2r' \underbrace{\langle\hat{O}\cos\sqrt{8}\phi(r')\rangle_{H_0}}_{0} + \frac{1}{2}\left(\frac{2g}{(2\pi\alpha)^2 u}\right)^2 \int d^2r' d^2r'' \langle\hat{O}\cos\sqrt{8}\phi(r')\cos\sqrt{8}\phi(r'')\rangle_{H_0}\right\}$$

$$\cong \left\{1 - \frac{1}{2}\left(\frac{2g}{(2\pi\alpha)^2 u}\right)^2\int d^2r'\int d^2r''\langle\cos\sqrt{8}\phi(r')\cos\sqrt{8}\phi(r'')\rangle_{H_0}\right\}$$

$$\times\left\{\langle\hat{O}\rangle_{H_0} + \frac{1}{2}\left(\frac{2g}{(2\pi\alpha)^2 u}\right)\iint_{d^2r'd^2r''}\langle\hat{O}\cos\sqrt{8}\phi(r')\cos\sqrt{8}\phi(r'')\rangle_{H_0}\right\} \simeq$$

$$\simeq \langle \hat{O} \rangle_{H_0} + \frac{1}{2}\left(\frac{2g}{(2\pi\alpha)^2 u}\right)^2 \int d^2r' \int d^2r'' \left\{ \begin{array}{l} \langle \hat{O} \cos\sqrt{8}\phi(r') \cos\sqrt{8}\phi(r'') \rangle_{H_0} \\ -\langle \hat{O} \rangle_{H_0} \langle \cos\sqrt{8}\phi(r') \cos\sqrt{8}\phi(r'') \rangle_{H_0} \end{array} \right\}$$

$$= \langle \hat{O} \rangle_{H_0} + \frac{1}{2}\left(\frac{g}{(2\pi\alpha)^2 u}\right)^2 \int d^2r' \int d^2r'' \sum_{\varepsilon_1=\pm 1} \sum_{\varepsilon_2=\pm 1} \left[\begin{array}{l} \langle \hat{O} e^{i\varepsilon_1\sqrt{8}\phi(r')} e^{-i\varepsilon_2\sqrt{8}\phi(r'')} \rangle_{H_0} \\ -\langle \hat{O} \rangle_{H_0} \langle e^{i\varepsilon_1\sqrt{8}\phi(r')} e^{-i\varepsilon_2\sqrt{8}\phi(r'')} \rangle_{H_0} \end{array} \right]$$

· Recall (p. 27) that

$$\langle e^{i\sum_j A_j \phi(r_j)} \rangle = e^{\frac{1}{2}\sum_{i\neq j} A_i A_j K F_1(r_i - r_j)} \qquad \left(\begin{array}{l}\text{with } \sum_j A_j = 0 \\ \text{otherwise zero}\end{array}\right)$$

Hence only $\varepsilon_1 = \varepsilon_2$ survives and it is:

$$\langle e^{ia\sqrt{2}\phi(r_1)} e^{-ia\sqrt{2}\phi(r_2)} e^{i\varepsilon_1\sqrt{8}\phi(r')} e^{-i\varepsilon_1\sqrt{8}\phi(r'')} \rangle =$$

$$= e^{\frac{1}{2}\left\{-2a^2 K F_1(r_1-r_2) + 4\varepsilon_1 a K F_1(r_1-r') - 4\varepsilon_1 a K F_1(r_1-r'') - 4\varepsilon_1 a K F_1(r_2-r') + 4K\varepsilon_1 a F_1(r_2-r'') - 8\varepsilon_1^2 K F_1(r'-r'')\right\}}$$

$$= \underbrace{e^{-a^2 K F(r_1-r_2)} e^{-4KF(r'-r'')}}_{\langle \hat{O} \rangle_{H_0} \langle e^{i\sqrt{8}\varepsilon_1\phi(r')} e^{-i\sqrt{8}\varepsilon_1\phi(r'')} \rangle} e^{2\varepsilon_1 a K\left[F_1(r_1-r') - F_1(r_1-r'') - F_1(r_2-r') + F_1(r_2-r'')\right]}$$

Hence:

$$\langle \hat{O} \rangle_H = \langle \hat{O} \rangle_{H_0} + \frac{1}{2}\left(\frac{2g}{(2\pi\alpha)^2 u}\right)^2 \int d^2r' \int d^2r'' \sum_{\varepsilon_1=\pm 1} e^{-a^2 K F_1(r_1-r_2)} e^{-4KF_1(r'-r'')}$$
$$\times \left\{ e^{2a\varepsilon_1 K(F_1(r_1-r') - F_1(r_1-r'') - F_1(r_2-r') + F_1(r_2-r''))} - 1 \right\}$$

$$= e^{-a^2 K F(r_1-r_2)} \times \left\{ \begin{array}{l} 1 + \frac{g^2}{2(2\pi\alpha)^4 u^2} \int d^2r' \int d^2r'' \sum_{\varepsilon_1=\pm 1} e^{-4KF_1(r'-r'')} \\ \times \left\{ e^{2a\varepsilon_1 K(F_1(r_1-r') - F_1(r_1-r'') - F_1(r_2-r') + F_1(r_2-r''))} - 1 \right\} \end{array} \right\}$$

* let $\left.\begin{array}{l} R = (r'+r'')/2 \\ r = r'-r'' \end{array}\right\}$:

$$R(r_1-r_2) = e^{-a^2 K F(r_1-r_2)} \left\{ \begin{array}{l} 1 + \frac{g^2}{2(2\pi\alpha)^4 u^2} \int d^2R \int d^2r \sum_{\varepsilon_1=\pm 1} e^{-4KF_1(r)} \\ \times \left[e^{2a\varepsilon_1 K\left[F_1(r_1-R-r/2) - F_1(r_1-R+r/2) - F_1(r_2-R-r/2) - F_1(r_2-R+r/2)\right]} - 1 \right] \end{array} \right\}$$

• Now, note that we can take a as small as we want.
Note also that $e^{-4\pi F(r)}$ is essentially a power law.
Hence the integral $\int d^2R \int d^2r$ is dominated by small r.
We can hence expand:

$$\sum_{\varepsilon_1}\left[e^{2a\varepsilon_1 K\left[F_1(r_1-R-r/2)-F_1(r_1-R+r/2)-F_1(r_2-R-r/2)-F_1(r_2-R+r/2)\right]}-1\right]\simeq$$

$$\simeq \sum_{\varepsilon_1=\pm1}\left\{e^{2a\varepsilon_1 K\left[-r\cdot\nabla F_1(r_1-R)+r\nabla F_2(r_2-R)\right]}-1\right\}$$

$$\simeq 2\cdot\frac{1}{2}(2aK)^2\left[r\cdot\nabla\left[F_1(r_1-R)-F_1(r_2-R)\right]\right]^2$$

Hence:

$$\mathcal{Z}(r_1-r_2)=e^{-a^2K F(r_1-r_2)}\left\{1+\frac{2g^2}{(2\pi a)^4u^2}\int d^2R\int d^2r\; e^{-4K F_1(r)}\times\left[aK\,r\cdot\nabla\left[F_1(r_1-R)-F_2(r_2-R)\right]\right]^2\right\}$$

→ Note that this contains terms of the form
$r_i r_j \nabla_{R_i}\left[F_1(r_1-R)-F_2(r_2-R)\right]\nabla_{R_j}\left[F_1(r_1-R)-F_2(r_2-R)\right]$

* We will introduce now a cut-off procedure (this time in real space) which corresponds to restricting $r>\alpha$.
(Note: this choice makes use of the rotational invariance of H_0 on the $(x, y=u\tau)$ plane. Recall that the asymptotic properties are independent of the short distance cut-off.)
• Since the cut-off preserves the symmetry $x\leftrightarrow -x$, $y\leftrightarrow -y$, then terms with $r_i r_j$ with $i\neq j$ are obviously zero,
and in addition $\int d^2r\, x^2=\int d^2r\, y^2=\int d^2r\,\frac{r^2}{2}$

· Hence:

$$\Omega(r_1-r_2) = e^{-a^2 K F(r_1-r_2)} \left\{ 1 + \frac{g^2 a^2 K^2}{(2\pi\alpha)^4 u^2} \int d^2R \int d^2r \; r^2 e^{-4KF_1(r)} \times \left[\left(\nabla_X [F_1(r_1-R) - F_1(r_2-R)] \right)^2 + \left(\nabla_Y [F_1(r_1-R) - F_1(r_2-R)] \right)^2 \right] \right\}$$

$=$ ← we integrate by parts $\Rightarrow \int d^2R \, (\nabla_X F)^2 = \underbrace{\int d^2R \, \nabla_X (F \nabla_X F)}_{0} - \int d^2R \, F \nabla_X^2 F$

$$= e^{-a^2 K F(r_1-r_2)} \left\{ 1 + \frac{g^2 a^2 K^2}{(2\pi\alpha)^4 u^2} \int d^2R \int d^2r \; r^2 e^{-4KF_1(r)} \times [F_1(r_1-R) - F_1(r_2-R)] (\nabla_X^2 + \nabla_Y^2) [F_1(r_1-R) - F_1(r_2-R)] \right\}$$

* Recall that $F_1(r)$ is essentially a log $\left(F_1(r) \sim \log(r/\alpha) \text{ when } r \gg \alpha \right)$

We can then use a quite useful expression

$$(\nabla_X^2 + \nabla_Y^2) \log(R) = 2\pi \delta(R)$$

(Note: this relation is well known from 2D Coulomb systems, where it just simply states that $\log(R)$ is the corresponding Green's function of the Laplace equation.)

Hence we may write:

$$(\nabla_X^2 + \nabla_Y^2) [F_1(r_1-R) - F_1(r_2-R)] \simeq 2\pi [\delta(r_1-R) - \delta(r_2-R)]$$

* Hence:

$$\int d^2R \, [F_1(r_1-R) - F_1(r_2-R)] (\nabla_X^2 + \nabla_Y^2) [F_1(r_1-R) - F_1(r_2-R)]$$

$$\simeq \int d^2R \, (F_1(r_1-R) - F_1(r_2-R)) \cdot 2\pi (\delta(r_1-R) - \delta(r_2-R))$$

· Note that once we have a cut-off at short distances $F_1(r_1-r_1)$ doesn't diverge (as it would without cutoff). Actually $F_1(r_1-r_1) \to F_1(\alpha) = 0$ (recall p. (25)).

- Hence:

$$\int d^2R\,[F_1(r_1-R)-F_1(r_2-R)]\,(\nabla_x^2+\nabla_y^2)\,(F_1(r_1-R)-F_1(r_2-R))$$

$$= -2\pi F_1(r_1-r_2) - 2\pi F_1(r_2-r_1) = -4\pi F_1(r_1-r_2)$$

- Hence:

$$R(r_1-r_2) = e^{-a^2K F_1(r_1-r_2)}\left\{1+\frac{g^2K^2a^2F_1(r_1-r_2)}{4\pi^3u^2\alpha^4}\int_{r>\alpha} d^2r\, r^2 e^{-4KF_1(r)}\right\}$$

$$\simeq e^{-a^2F_1(r_1-r_2)\left[\cancel{K}-\frac{g^2K^2}{4\pi^3u^2\alpha^4}\int_{r>\alpha} d^2r\, r^2 e^{-4KF_1(r)}\right]}$$

$$= e^{-a^2K_{eff}F_1(r_1-r_2)}$$

- This has the same form as $\langle \hat{O}\rangle_{H_0}$, i.e. as $R(r_1-r_2)$ calculated with H_0, but now we have an effective exponent:

$$F_1 = \frac{1}{2}\ln(r^2/\alpha^2)$$

$$K_{eff} = K - \frac{g^2K^2}{2(\pi u)^2\alpha^4}\int_{r>\alpha} dr\, r^3 e^{-4KF_1(r)}$$

$$= K - \frac{g^2K^2}{2(\pi u)^2\alpha^4}\int_{r>\alpha} dr\, r^3\left(\frac{r}{\alpha}\right)^{-4K}$$

$$= K - \frac{g^2K^2}{2(\pi u)^2}\int_{\alpha}^{\infty} d\left(\frac{r}{\alpha}\right)\left(\frac{r}{\alpha}\right)^{3-4K}$$

- This effective exponent of $R(r_1-r_2)$ is precisely what controls the asymptotic (low energy) properties of the system, and hence should remain unaffected by the cut-off.

* Let's consider a shift in the cut-off:

$$K_{eff} = K^{(\alpha)} - \frac{g^2 K^2}{2(\pi u)^2} \int_{\alpha}^{\alpha+d\alpha} d\left(\frac{r}{\alpha}\right)\left(\frac{r}{\alpha}\right)^{3-4K} - \frac{g^2 K^2}{2(\pi u)^2} \int_{\alpha+d\alpha}^{\infty} d\left(\frac{r}{\alpha}\right)\left(\frac{r}{\alpha}\right)^{3-4K}$$

$$\cong K^{(\alpha)} - \frac{g^2 K^2}{2(\pi u)^2}\frac{d\alpha}{\alpha} - \frac{g^2 K^2}{2(\pi u)^2}\left(\frac{\alpha+d\alpha}{\alpha}\right)^{4-4K} \int_{\alpha+d\alpha}^{\infty} d\left(\frac{r}{\alpha+d\alpha}\right)\left(\frac{r}{\alpha+d\alpha}\right)^{3-4K}$$

Let $K(\alpha+d\alpha) = K(\alpha) - \frac{g^2_{(\alpha)} K(\alpha)^2}{2(\pi u)^2}\frac{d\alpha}{\alpha}$

$$g^2(\alpha+d\alpha) = g^2(\alpha)\left(\frac{\alpha+d\alpha}{\alpha}\right)^{4-4K}$$

Then:

$$K_{eff} \simeq K(\alpha+d\alpha) - \frac{g(\alpha+d\alpha)^2 K(\alpha+d\alpha)^2}{2(\pi u)^2} \int_{\alpha+d\alpha}^{\infty} d\left(\frac{r}{\alpha+d\alpha}\right)\left(\frac{r}{\alpha+d\alpha}\right)^{3-4K(\alpha+d\alpha)}$$

and K_{eff} keeps unchanged.

* ~~Hence~~ Let's take as usual $\alpha = \alpha_0 e^{\ell}$ where α_0 is the original cut-off.

Then: $\frac{d\alpha}{\alpha} = d\ell$:

$$K(\ell+d\ell) = K(\ell) - \frac{g(\ell)^2 K(\ell)^2}{2(\pi u)^2} d\ell$$

$$g(\ell+d\ell) = g(\ell)\, e^{(2-2K)d\ell} \longrightarrow g(\ell+d\ell) = g(\ell) + g(\ell)(2-2K)\,d\ell$$

Hence:

$$\boxed{\begin{aligned} \frac{dK(\ell)}{d\ell} &= -\frac{g(\ell)^2 K^2(\ell)}{2(\pi u)^2} \\ \frac{dg(\ell)}{d\ell} &= (2-2K(\ell))\,g(\ell) \end{aligned}} \rightarrow \text{Renormalization group eqs.}$$

Note: the RG equations of p. 58 and those of pages 50 and 51 are equivalent in the sense that they give the same asymptotic behaviour, as it should because the way we take the cut-off should be irrelevant. For the following discussion we shall employ the RG eqs. of p. 58.

- Physical meaning of the RG equations

- let's try to understand the RG equations on a physical basis.
- The cosine term in the action is

$\sim g \int dx \int d\tau \cos(\sqrt{8}\phi)$

since $\langle \cos\sqrt{8}\phi_\sigma(r) \cos\sqrt{8}\phi_\sigma(0) \rangle \sim \left(\frac{\alpha}{r}\right)^{4K}$

then $\cos\sqrt{8}\phi_\sigma(r) \sim L^{-2K}$ (L = length)

Hence $g \int dx \int d\tau \cos\sqrt{8}\phi \sim g L^{2-2K}$

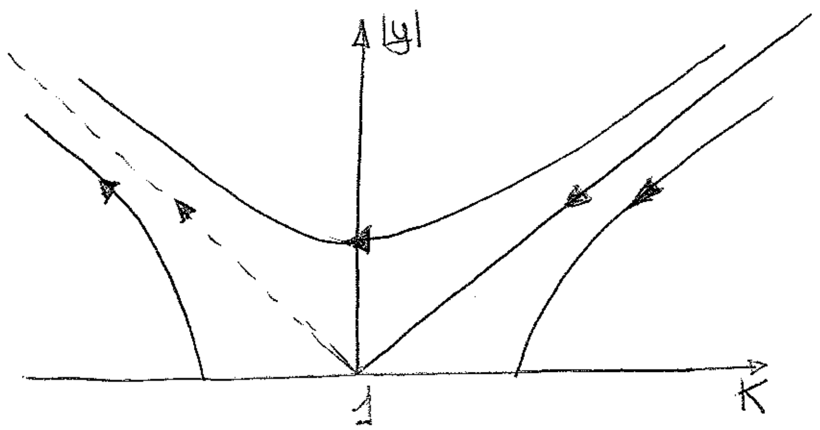
Changing $\left.\begin{array}{l} L \to L + dl \\ g \to g + dg \end{array}\right\}$ $(g+dg)(L+dl)^{2-2K} \cong g L^{2-2K}\left(1 + dg/g + (2-2K)\frac{dl}{L}\right)$

and hence $dg/dl \sim (2-2K)\, g$

- Hence the equation for $\frac{dg}{dl}$ is a mere question of scaling.
- let's try now to grasp the idea behind the other equation. Note that the term $\frac{u}{K}(\nabla\phi_\sigma)^2$ acts as a dispersion for the field ϕ_σ. On the other side we have the $\cos\sqrt{8}\phi_\sigma$ term that wants to order the field ϕ_σ. Hence when "put into a harmonic form" the $\cos\sqrt{8}\phi_\sigma$ tends to reduce K, i.e. to reduce dispersion / favouring order (note the minus sign in the equation for dK/dl).

- RG-flow for the sine-Gordon Hamiltonian

* The RG equations of p. 58 lead to the following RG-flow diagram ($y = g/\pi u$)



- For an infinitesimal y, one sees from the flow diagram that:
  - For $K>1$: y is irrelevant
  - For $K<1$: y is relevant

(- Note: in the RG-language relevant means that it grows upon a change of scale, whereas irrelevant means that it decreases upon a change of scale)

- One thus expects a phase transition at $K=1$.

Let's analyze what happens close to this transition.

Let $K = 1 + y_{\parallel}/2$

(Note: assuming $g_{2\parallel} = g_{2\perp}$ and $g_{4\parallel} = g_{4\perp}$, then $K_\sigma \simeq 1 + y_{\parallel}/2$ p. 37)

- Then: $\left.\begin{aligned} \frac{dy_{\parallel}}{dl} &= -y^2(l) \\ \frac{dy}{dl} &= -y_{\parallel}(l)\,y(l) \end{aligned}\right\}$ $y\frac{dy}{dl} = -y_{\parallel} y^2 = +y_{\parallel}\frac{dy_{\parallel}}{dl}$ $\Longrightarrow \frac{d}{dl}(y_{\parallel}^2 - y^2) = 0$

Hence $y_{\parallel}^2 - y^2 = A^2 = \text{constant} \Longrightarrow$ These are hyperbolas (as depicted in the figure above)

(Note: $A^2>0$ below the lines $y_{\parallel} = \pm|y|$, and $A^2<0$ over those lines.)

- Note also that the RG eqs only depend on $|y|$. In the following we consider $y>0$.

- It's clear that the line $y_{\parallel} = y$ is a separatrix between 2 regimes: for $y_{\parallel} > y$ there's a flow towards $|y| = 0$, and hence y is irrelevant, whereas for $y_{\parallel} < y$ there's a flow towards $|y| = \infty$, and hence y is relevant.
- Let's have a look first to the case $y_{\parallel} > y$ (massless regime)

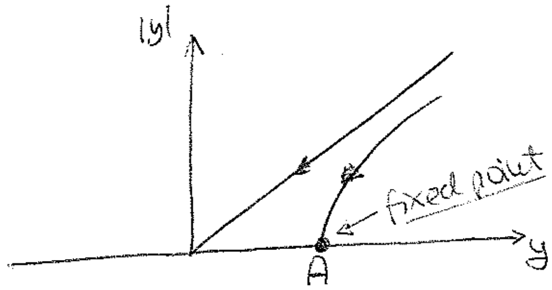
For $y_{\parallel} > y \longrightarrow A^2 > 0$, and we may take $y_{\parallel} = \frac{A}{\text{th}\,\gamma}$, $y = \frac{A}{\text{sh}\,\gamma}$

(Note: $\frac{1}{\text{th}^2\gamma} - \frac{1}{\text{sh}^2\gamma} = 1$)

Hence $\frac{dy}{dl} = -A \frac{d\gamma}{dl} \frac{\text{ch}\,\gamma}{\text{sh}^2\gamma} = -y_{\parallel} y = -A^2 \frac{1}{\text{th}\gamma\,\text{ch}\gamma} \Longrightarrow \frac{d\gamma}{dl} = A$

$\Longrightarrow \gamma = \gamma_0 + A(l - l_0)$

Thus $\gamma = Al + \text{atanh}\left(\frac{A}{y_{\parallel}^0}\right)$



Hence: $y_{\parallel}(l) = \dfrac{A}{\tanh\left(Al + \text{atanh}(A/y_{\parallel}^0)\right)}$

$$y(l) = \frac{A}{\sinh\left(Al + \text{atanh}(A/y_{\parallel}^0)\right)}$$

- On the separatrix $y_{\parallel} = y \longrightarrow \frac{dy_{\parallel}}{dl} = \frac{dy}{dl} = -y^2 \longrightarrow y_{\parallel} = y = \frac{y_0}{1 + y_0 l}$

- The fixed point (i.e. where the flow ends) is hence $y^* = 0$, $y_{\parallel}^* = A$

Close to the fixed point: $dy_{\parallel}/dt \simeq 0$ ← the trajectories are nearly vertical

$dy/dt \simeq (2 - 2K^*)\,y$ — K converges to a fixed point value K^*

The correlation functions may be then computed using the quadratic Hamiltonian but with the renormalized K^* ($K_{eff} = K^*$)

$$\boxed{\left\langle e^{ia\sqrt{2}\phi_\sigma(r)} e^{-ia\sqrt{2}\phi_\sigma(0)} \right\rangle \simeq \left(\frac{a}{r}\right)^{a^2 K^*}}$$

. Hence in the regime $y_{\parallel} > y$ the RG allows ~~us~~ to compute the asymptotic behavior of the correlation functions, because the cosine term disappears and we recover a purely harmonic Hamiltonian. This regime is the so-called massless regime

- On the separatrix ($y_{\parallel} = y$) the cosine term is marginally irrelevant

Note: in the RG jargon marginally irrelevant is the intermediate regime (i.e. in our case the separatrix.) between relevant and irrelevant character,

Note that $y_{\parallel} = y$ means rotation invariance (spin-independent interactions), since the flow is along the diagonal this rotational invariance is preserved all the way till the fixed point $y_{\parallel}^{*} = y^{*} = 0$ ($K^{*} = 1$). (Note: recall from p. (62) that $K_{\sigma} = 1$ guaranteed spin-rotation symmetry)

However the fact that the cosine is only marginally irrelevant leads to logarithmic corrections (which we will not discuss now) in the correlation functions

Luther-Emery liquid
$\Uparrow$
- Let's have a look now to the $y > y_{\parallel}$ case (massive regime)

In this region, the flow tends to $y_{\parallel} \to -\infty$ and $|y| \to \infty$,

i.e. the flow goes to strong coupling.

Note that the RG equations have been established assuming a small coupling, a perturbation expansion which obviously is not valid when $y \sim 1$.

. In the regime $y < 1$ we may still employ the RG solutions.

let $\bar{A} = \sqrt{y_0^2 - (y_{\parallel}^0)^2}$. In the expressions of p. (61) we employ $A = i\bar{A}$,

and use that $\tanh i\gamma = i \tan\gamma$, to obtain:

$$\arctan\left(\frac{y_{\parallel}^{0}}{\bar{A}}\right) - \arctan\left(\frac{y_{\parallel}}{\bar{A}}\right) = \bar{A}\ell$$

- When $\bar{A}=0$ (i.e on the line $y_{\parallel}=-y<0$) we have

$$y(l) = \frac{y_0}{1-y_0 l}$$

- Since the flow goes to strong coupling (where the perturbative RG treatment isn't valid) we will have to "guess" the physics in this regime.
- When $|y| \to \infty$ the cosine term $\frac{2g}{(2\pi\alpha)^2}\int dx \cos(\sqrt{8}\phi_\sigma)$ is very strong, and hence imposes that ϕ_σ must be locked in a minimum of the cosine → i.e. the field orders → we go into a <u>massive phase</u>
- When $|y|$ is very large we can expand hence around the minimum. For $y\to-\infty$ the minimum is $\phi_0=0$ (for $y\to\infty$ it would be $\phi_0=\pi$).

Expanding, we get (recall $y=g/\pi u$)

$$H = H^0 + \frac{2yu}{\pi\alpha^2}\int dx\, \phi_\sigma^2(x)$$

Hence $S = S^0 + \int d\tau \int dx \frac{2yu}{\pi\alpha^2}\phi_\sigma^2(x) \overset{\text{Fourier transforming the action}}{=}$

$$= \frac{1}{2\pi K}\frac{1}{\beta\Omega}\sum_{k,\omega_n}\left[\frac{1}{u}\omega_n^2 + \underbrace{uk^2 + \frac{4Kyu}{\alpha^2}}\right]\phi^*(k,\omega_n)\phi(k,\omega_n)$$

Note that the eigenenergies are now of the form

$$\epsilon(k)^2 = u^2k^2 + \frac{4Kyu^2}{\alpha^2}$$

and hence the system presents a <u>gap</u>: $\boxed{M_\rho = \sqrt{\frac{4Kyu^2}{\alpha^2}}}$

<u>These excitations are hence now massive.</u> They correspond to the variation of ϕ_σ within one of the minima of the cosine.

(<u>Note</u>: there are other excitations (<u>solitons</u>) that take ϕ from one minimum to another, but we won't see them by now).

- The previous result is just valid if $|y|$ is very large.

Let's analyze now ~~how~~ how the flow approaches the large $|y|$ regime.

- Let's define ℓ^* such that $y(\ell^*) \sim 1$.

From the large-y result we see that the gap scales as

$$\Delta_\sigma(\ell^*) \sim u\, y^{1/2}(\ell^*)/\alpha \sim u/\alpha = \Delta_0$$

Since the gap in the spectrum has units of energy it renormalizes as $\Delta_\sigma(\ell) = e^{\ell}\,\Delta_\sigma(\ell=0)$

(Note: the energy is extensive and hence it scales as the length, i.e. as e^{ℓ})

Hence, the true gap of the system is simply given by

$$\Delta_\sigma(\ell=0) \sim e^{-\ell^*}\Delta_0$$

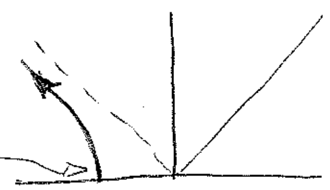
- Let's evaluate ℓ^* and hence Δ_σ for different regimes:

  - $y \ll |y_{\parallel}|$

The RG eqs become

$$\frac{dK}{d\ell} \simeq 0$$

The flow is nearly vertical



$$\frac{dy}{d\ell} = (2-2K)\,y(\ell)$$

Hence $y(\ell) = y_0\, e^{(2-2K)\ell} \longrightarrow y(\ell^*) = 1 \Rightarrow e^{(2-2K)\ell^*} = \frac{1}{y_0}$

$$\longrightarrow e^{\ell^*} = \left(\frac{1}{y_0}\right)^{\frac{1}{2-2K}}$$

Thus: $$\boxed{\frac{\Delta_\sigma(\ell=0)}{\Delta_0} \sim (y_0)^{\frac{1}{2-2K}}}$$

The gap is a power law of the bare coupling constant. When $K \to 1$ the gap gets smaller and smaller (recall that we assume $y_0 \ll 1$)

• $y_{\parallel} = -y \implies$ This corresponds to a system invariant by spin rotation (we can take $y = -y_{\perp\perp}$ and $y_{\parallel\parallel} = y_{\perp\perp} < 0$)

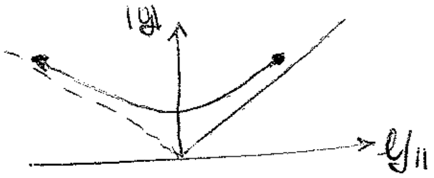
Recall from p. 63 that

$$y(\ell) = \frac{y_0}{1-y_0 \ell} \longrightarrow y(\ell^*) = \frac{y_0}{1-y_0\ell^*} = 1 \Rightarrow \ell^* = \frac{1}{y_0} - 1 \underset{y_0 \ll 1}{\simeq} 1/y_0$$

Hence $\boxed{\Delta_\sigma(\ell=0) \simeq \Delta_0 e^{-1/y_0}}$

The gap is hence <u>exponentially small</u> in the coupling constant.

* let's see finally what happens <u>close to the transition</u> between massive and massless regime, i.e. close to the separatrix $y = y_{\parallel}$ (but slightly over it)



In the $y > y_{\parallel}$ region (p. 64):

$$\arctan\left(\frac{y_{\parallel}^0}{\bar{A}}\right) - \arctan\left(\frac{y_{\parallel}}{\bar{A}}\right) = \bar{A}\ell$$

Since at the separatrix $\bar{A} = \sqrt{y_0^2 - y_{\parallel}^{0\,2}} \longrightarrow 0$, and since $y_{\parallel}^0 > 0$ and $y_{\parallel}(\ell^*) = -y(\ell^*) < 0 \implies y_{\parallel}^0/\bar{A} \longrightarrow +\infty$ and $y_{\parallel}(\ell^*)/\bar{A} \longrightarrow -\infty$

Consequently: $\bar{A}\ell^* \simeq \pi \longrightarrow \ell^* \simeq \pi/\bar{A} \longrightarrow \Delta_\sigma(\ell=0) = \Delta_0 e^{-\pi/\bar{A}}$

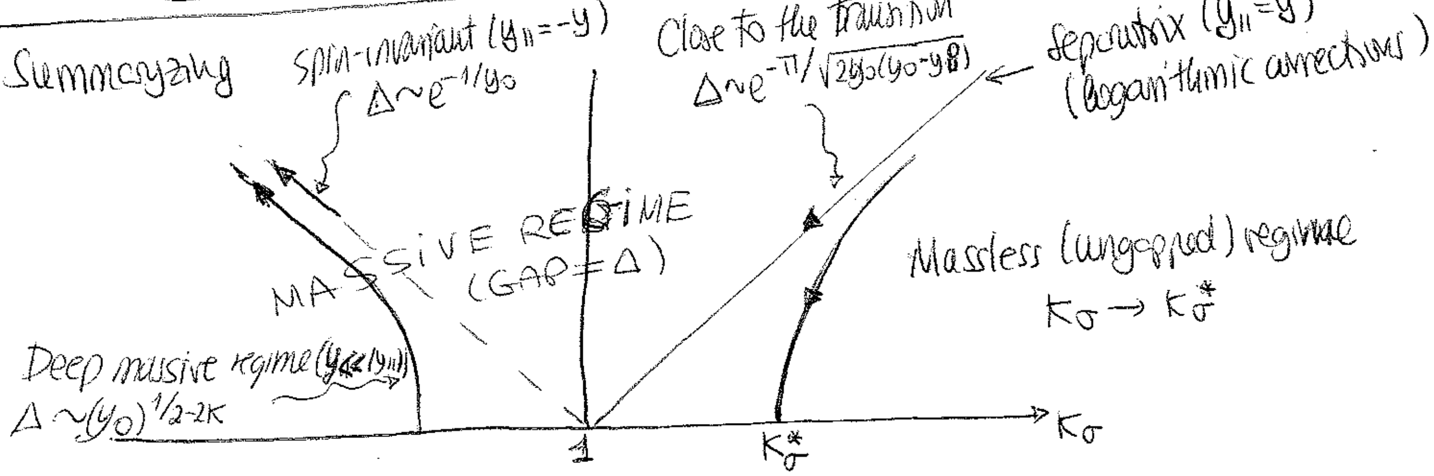
let's see what happens when y_0 approaches $y_{\parallel}^0$ very close:

$$\bar{A} = \sqrt{(y_0+y_{\parallel}^0)(y_0-y_{\parallel}^0)} \simeq \sqrt{2y_{\parallel}^0(y_0-y_{\parallel}^0)}$$

Hence $\boxed{\Delta_\sigma(\ell=0) \sim e^{-\pi/\sqrt{(y_0-y_{\parallel}^0)y_{\parallel}^0 2}}}$

The gap is exponentially small in the square root of the distance to the transition.

* Summarizing



* "Phase diagram" for spin-1/2 fermions

• Massless sector

. In the massless sector both charge and spin sector lead to a power-law decay of correlations (but with K_σ^*)
(Note: for spin rotation invariant interactions, the massless sector $g_{\parallel} > g$ means $g_1 > 0$, i.e. repulsive interactions).

. Recall our discussion on O_{CDW} and $O_{SDW}^{x,y,z}$:

$$\langle O_{CDW}^{\dagger}(\vec{r})\, O_{CDW}(0)\rangle \sim \langle O_{SDW}^{z\dagger}(\vec{r})\, O_{SDW}^{z}(0)\rangle \sim \left(\alpha/r\right)^{K_\rho + K_\sigma^*}$$

$$\langle O_{SDW}^{x,y\dagger}(\vec{r})\, O_{SDW}^{x,y}(0)\rangle \sim \left(\alpha/r\right)^{K_\rho + 1/K_\sigma^*}$$

Similarly:

$$\langle O_{SS}^{\dagger}(\vec{r})\, O_{SS}(0)\rangle \sim \langle O_{TS}^{z\dagger}(\vec{r})\, O_{TS}^{z}(0)\rangle \sim \left(\alpha/r\right)^{1/K_\rho + K_\sigma^*}$$

$$\langle O_{TS}^{x,y\dagger}(\vec{r})\, O_{TS}^{x,y}(0)\rangle \sim \left(\alpha/r\right)^{1/K_\rho + 1/K_\sigma^*}$$

* For a system with rotation symmetry $K_\sigma^* = 1$ and the dominant instabilities (recall our discussion of p. 31) are
  - For $K_\rho < 1$ (repulsive interactions): SDW and CDW
  - For $K_\rho > 1$ (attractive interactions): SS and TS

* Note that (as far as the power law is concerned) there's as much tendency to SDW_z as to CDW when the interactions are repulsive. That's somehow counterintuitive, since one would expect that the repulsion between fermions should handicap the CDW, i.e. we would expect antiferromagnetic tendency (SDW_z) but not charge density tendency.
  In fact this is true at short distances, but at large distances (and this is what we analyze with the power laws) the spin up and down density waves are soft enough (note that SDW_z means that the ↑ and ↓ density waves are in counter-phase and CDW means that they are in phase).
  Note also that the amplitudes aren't the same (actually even log. corrections favor the SDW against CDW)

• Massive sector (Luther-Emery liquid)

. For a spin rotation invariant problem, this occurs along $y_{\parallel} = -|y|$, hence $g_{1\parallel} = g_{1\perp} = g_1 < 0$ → attractive interactions.

. For a general case that it is not spin rotation invariant we can have a massive phase because $K_\sigma < 1$ while $g_{1\perp}$ may be >0 or <0.

. Recall that in the massive sector ϕ_σ orders in the value that minimizes the cosine term $g_{1\perp} \int dx \cos\sqrt{8}\phi(x)$.

If $g_{1\perp} > 0 \longrightarrow \phi_\sigma = \frac{\pi}{\sqrt{8}} + \frac{2\pi n}{\sqrt{8}}$

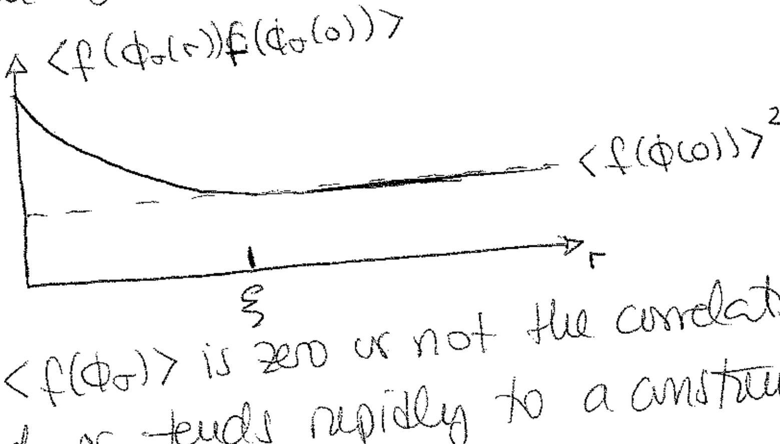
$g_{1\perp} < 0 \longrightarrow \phi_\sigma = 0 + \frac{2\pi n}{\sqrt{8}}$

and the field is trapped in one of these minima.

. Going from the massless phase to the massive one is hence a true quantum phase transition driven by K_σ (which control the amount of fluctuations from the quadratic part)

. When a field orders (in this case ϕ_σ), the correlation functions containing that field decay exponentially, with a characteristic length (correlation length) given by the inverse of the gap:

$$\boxed{\xi = u/\Delta}$$



Depending on whether $\langle f(\phi_\sigma) \rangle$ is zero or not the correlation is either asymptotically suppressed or tends rapidly to a constant.

. Note also that since ϕ_σ orders, the dual field θ_σ is totally disordered $\longrightarrow \langle \cos(a\theta_\sigma) \rangle = \langle \sin(a\theta_\sigma) \rangle = 0$, and all correlation functions involving it decay exponentially to zero. Recall from pages 43, 44 and 45 that this is the case of $SDW_{x,y}$ and $TS_{x,y}$.

• On the contrary CDW, SDW_z, SS and TS_z depend on ϕ_σ. Whether they tend to zero or not depends on the value of the locked ϕ_σ.

* If $g_{1\perp} < 0$:

• In that case $\phi_\sigma = 0 + 2n\pi/\sqrt{8} \rightarrow \langle \sin\sqrt{2}\,\phi_\sigma \rangle = 0$

Hence SDW_z (p. 42) and TS_z (p. 45) decay to zero.

• The exponential suppression of SDW_z and SS is intuitively reasonable, since for attractive interactions between opposite spins ($g_{1\perp} < 0$) one expects the fermions to pair in singlets, and hence any spin correlation is exponentially suppressed. Accordingly all SDW_{xyz} and TS_{xyz} fluctuations are exponentially suppressed.

• In the CDW and SS correlations the spin part can be replaced by a constant and the asymptotic decay goes as:

$$\langle O^{+}_{CDW}(\vec{r})\, O_{CDW}(0) \rangle \propto C^2 \left(\alpha/r\right)^{K_\rho}$$

$$\langle O^{+}_{SS}(\vec{r})\, O_{SS}(0) \rangle \propto C^2 \left(\alpha/r\right)^{1/K_\rho}$$

with $C = \langle \cos\sqrt{2}\,\phi_\sigma \rangle$ (if the field was really stuck at the minimum then $C = 1$, but due to quantum fluctuations $C < 1$).

• Compared to the massless sector it's clear that the correlation functions for CDW and SS <u>decay more slowly</u>.

• Recall our discussion on <u>susceptibilities</u> in p. 31. Recall that for correlations decaying as $r^{-\nu}$ the corresponding susceptibility diverges if $\nu - 2 < 0$. Hence χ_{CDW} diverges if $K_\rho < 2$ and χ_{SS} if $K_\rho > 1/2$.

Hence between $1/2 \leq K_\rho \leq 2$ both susceptibilities diverge.

• If $K_\rho < 1/2$ (very strong repulsion) only χ_{CDW} is divergent, whereas if $K_\rho > 2$ (very attractive) only χ_{SS} is divergent

* If $g_{1\perp} > 0$

- In that case $\phi_\sigma = \pi/\sqrt{8} + 2n\pi/\sqrt{8} \longrightarrow \langle \cos\sqrt{2}\phi_\sigma \rangle = 0$

Hence CDW (p. 40) and SS (p. 45) decay to zero, whereas SDW$_z$ and TS$_z$ have correlations similar to those of CDW and SS for the $g_{1\perp} < 0$ case.

- Note that since $K_\sigma \simeq 1 + g_{1\parallel}/2\pi v_F$ then $K_\sigma < 1$ and $g_{1\perp} \gtrless 0$ means $g_{1\parallel} < 0$ and $g_{1\perp} > 0$ → very spin anisotropic interactions. It's hence not surprising that SDW$_z$-fluctuations dominate.

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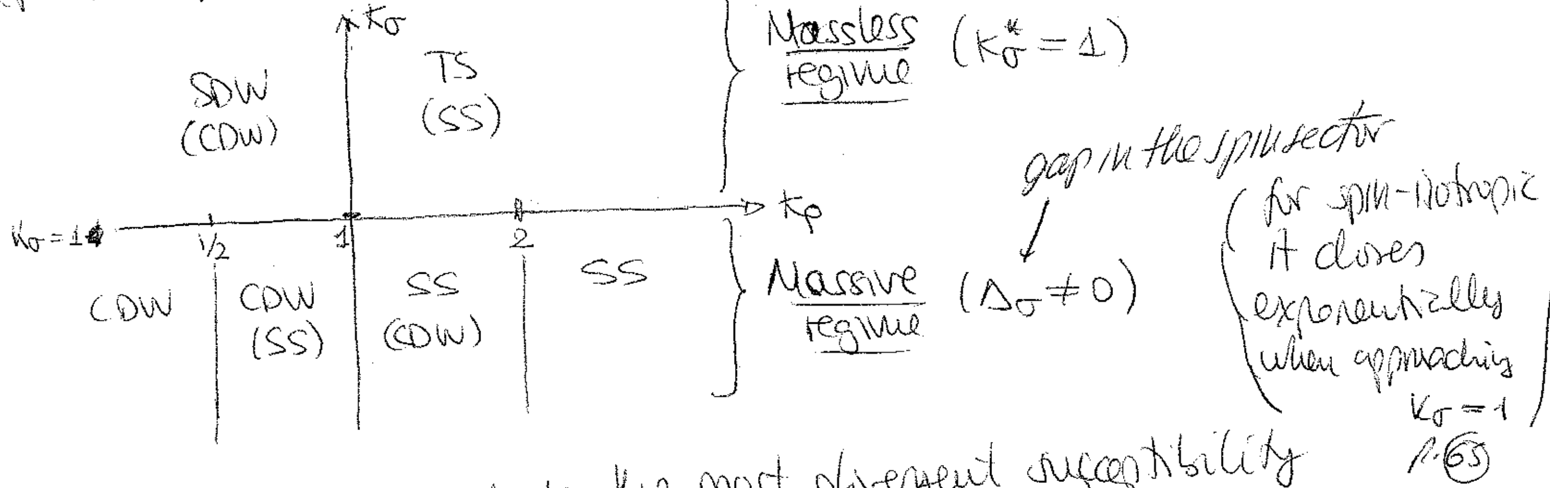
* Let's finally summarize with a phase diagram for the spin-1/2 system (for the spin-isotropic case)



- The phases correspond to the most divergent susceptibility whereas subdominant divergences are indicated in parenthesis. for the $K_\sigma > 1$ which one dominates is given by logarithmic corrections as discussed above).

- Let's finally note that any microscopic model with simple short-range interactions will be described by this phase diagram.