INTRODUCTION : 3D vs. 1D

* Fermi-liquid theory

• Recall that at $T=0$ all states of a Fermi system are occupied up to the Fermi energy. The occupation n_k has hence a discontinuity at the Fermi surface (k_F). This discontinuity is 1 for free electrons.
• The excitations have well defined momentum and energy $E(k)$, and they consist in adding particles. The excitations are eigenstates of the Hamiltonian and have hence an infinite lifetime.
• The remarkable result of Landau's Fermi liquid theory is that not much changes when the fermions interact with each other. The elementary particles are not anymore the individual fermions, but fermions dressed by density fluctuations (bosons), and hence the elementary excitations are fermions as well. These elementary excitations are called quasiparticles. They can be considered as free (although residual interactions may be relevant). The occupation number n_k is also discontinuous at the Fermi surface, but the amplitude of the discontinuity is $Z<1$.
($Z \approx$ fraction of the fermion remaining in the quasiparticle).
There's also a well defined relation $E(k)$ (of course different from the bare energy of the fermion) which close to the Fermi surface may be expanded as:

$$E(k) \cong E(k_F) + \frac{k_F}{m^*}(k-k_F)$$

effective mass

Close to the Fermi surface, the change in dispersion amounts only for the change $m \to m^*$

Since the volume enclosed by the Fermi surface is invariant against interactions (Luttinger theorem), for a spherical Fermi surface k_F remains unchanged.
Because they are not completely free the quasiparticles have also a lifetime τ (due to the scattering between quasiparticles), which becomes larger and larger when approaching k_F.

• This theory is valid even for very large interactions.

• In addition to the individual quasiparticles other types of excitations may occur. One can define collective excitations (disturbances of the density or spin density). Density waves in short-range-interacting Fermi gases build the so-called zero-sound, whereas plasmon-excitations are built when long-range interactions are present.

* Failure of Fermi-liquid theory in 1D

• As mentioned above the idea of nearly free quasiparticles is crucial for the Fermi-liquid theory. This is OK in dimensions larger than one, but not in 1D. In 1D a fermion that tries to propagate has to push its neighbors, and hence no individual motion is possible, only collective excitations are feasible (however weak the interaction). Fermi-liquid theory hence fails in 1D.

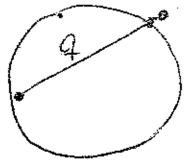
• We will need hence another type of theory, which we will discuss these lectures.

Particle-hole excitations

A crucial component of the excitations of a Fermi gas is the so-called particle hole excitations where an electron is taken from below the Fermi level and promoted above.

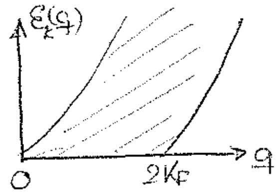
• Since one destroys a particle with momentum k and creates a particle with momentum q the momentum of the excitation is well fixed and equal to q. The energy of the excitation depends in general on k and on q.

• In dimensions larger than 1 one can create particle-hole excitations of arbitrary low energy for all $q < 2k_F$.

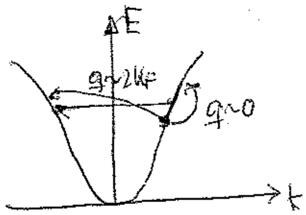


That's easy. One kills a particle just below some point of the Fermi surface and create somewhere else like in the figure.

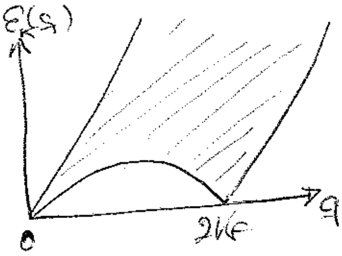
The particle-hole excitations thus lead to a continuum extending to zero energy for all $|q| < 2k_F$.



* In 1D the situation is indeed very different. The "Fermi surface" has just 2 points, and the only way to get low-energy excitations is to destroy and re-create pairs close to $q=0$ and $q=2k_F$



Hence the particle-hole spectrum is very different $\Longrightarrow$



* Let's focus on the behavior close to $q=0$. The energy of a particle-hole excitation is:

$$E_k(q) = \xi(k+q) - \xi(k) \qquad \text{with } \xi(k) = \frac{k^2 - k_F^2}{2m}$$

Then: close to k_F

$$\xi(k) = \frac{\left[(k-k_F)+k_F\right]^2 - k_F^2}{2m} = (k-k_F)\frac{k_F}{m} + \frac{(k-k_F)^2}{2m}$$

Hence:
$$E_k(q) = \frac{k_F}{m} q + \frac{(k-k_F)\,q}{m} + \frac{q^2}{2m}$$

For $q \approx 0$ the average $E(q)$ for $k \in [k_F - q, k_F]$ is

$$E(q) \approx \frac{k_F}{m} q = v_F q \qquad (v_F \equiv \text{Fermi velocity})$$

and the variance $\delta E(q) = \max(E_k(q)) - \min(E_k(q))$ is

$$\delta E(q) = E_{k_F - q}(q) - E_{k_F}(q) = \frac{q^2}{m}$$

* Hence, we have here two crucially important results:

(i) The average energy of a particle-hole excitation is only dependent on its momentum $q \Rightarrow E(q) = v_F\, q$
Hence particle-hole excitations are excitations with well defined momentum q and energy $E(q)$.

(ii) The dispersion $\delta E(q)$ goes as:

$$\delta E(q) = \frac{q^2}{m} = \frac{1}{m v_F^2} E(q)^2$$

and hence it goes to zero much faster than $E(q)$ when $q \to 0$. This means that the particle-hole excitations become longer and longer lived when $q \to 0$.

(Note: recall that width in energy means finite life time. Hence vanishing $\delta E(q)$ means very long life-time.)

. Note that the particle-hole excitations are made of the destruction and creation of a fermion, and they are hence <u>bosonic</u>.

. This crucial point is at the root of the bosonization method that we shall discuss in these lectures.