LUTTINGER LIQUIDS

- Recall (p. 5) that our derivation of the bosonization techniques is based on the linearization of the spectrum at the Fermi surface (Tomonaga-Luttinger model). But what happens when it isn't possible to remain confined at the Fermi surface, i.e. because the interactions are very strong?

In the following lecture we shall have a look to the idea of Luttinger liquid, which in 1D plays a similar role as that of Fermi liquids in higher dimensions.

Phenomenological bosonization

- In the following we shall re-derive the bosonization formulas in a phenomenological way, following ideas of Haldane.
- Let's consider any 1D system (fermions or bosons). The density operator of the system is:

$$\rho(x) = \sum_i \delta(x - x_i) \qquad x_i \rightarrow \text{position operator of the } i^{th} \text{ particle.}$$

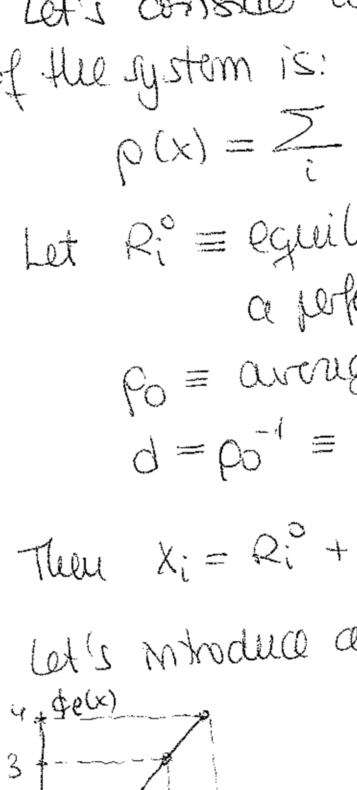
Let $R_i^0 \equiv$ equilibrium position that the particle would occupy in a perfect crystalline lattice

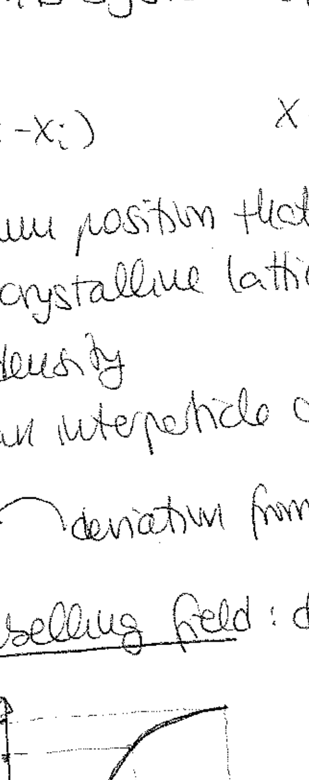
$\rho_0 \equiv$ average density

$d = \rho_0^{-1} \equiv$ mean interparticle distance $\longrightarrow R_i^0 = d\,i$

Then $x_i = R_i^0 + u_i$ ← deviation from the equilibrium position.

- Let's introduce a labelling field: $\phi_e(x)$, such that $\phi_e(x_i) = 2\pi i$





⟸ Clearly $\phi_e(x)$ is a way of numbering the particles, something one can always do in 1D in a unique way.

* Note that

$$\rho(x) = \sum_n \delta\left[x - \phi_e^{-1}(2\pi n)\right] = \quad \delta\left[\phi_e(x) - 2\pi n\right] = \frac{1}{|\nabla\phi_e(x)|}\delta\left[x - \phi_e^{-1}(2\pi n)\right]$$

[Note: This is because $\delta(f(x)) = \sum_{\text{zeros of } f} \frac{1}{|f'(x_i)|}\delta(x - x_i)$]

$$= \sum_n |\nabla\phi_e(x)|\, \delta\left[\phi_e(x) - 2\pi n\right] = \quad \phi_e \text{ grows monotonically, hence } \nabla\phi(x) \geq 0$$

$$= \nabla\phi_e(x) \sum_n \delta\left[\phi_e(x) - 2\pi n\right] = \quad \text{Poisson summation formula: } \sum_{n=-\infty}^{\infty} f(t+nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\tilde{f}\left(\frac{k}{T}\right)e^{2\pi i k t/T}$$

(Note: Here $T = 2\pi$, $f(x) = \delta(x) \to \tilde{f}(k) = 1$

$$= \nabla\phi_e(x)\, \frac{1}{2\pi}\sum_p e^{ip\phi_e}$$

* We introduce now a field $\phi(x)$ relative to the perfect crystalline

solution: $\phi_e(x) = 2\pi\rho_0 x - 2\phi(x) \longrightarrow \nabla\phi_e(x) = 2\pi\rho_0 - 2\nabla\phi$

Hence $$\boxed{\rho(x) = \left(\rho_0 - \frac{1}{\pi}\nabla\phi\right)\sum_p e^{i2p\left[\pi\rho_0 x - \phi(x)\right]}}$$

Since the density operator commutes with itself so we expect for $\phi(x)$.

* Note that if we average $\rho(x)$ over distances $\gg 1/\rho_0$ then only the $p=0$ term survives and we may approximate

$$\rho(x) \simeq \rho_0 - \frac{1}{\pi}\nabla\phi(x)$$

* Let's have a look to the single-particle operator, which we may always write in the density-phase representation as

$$\left.\begin{aligned} \psi^\dagger(x) &= [\rho(x)]^{1/2} e^{-i\Theta(x)} \\ \psi(x) &= e^{i\Theta(x)}[\rho(x)]^{1/2} \end{aligned}\right\} \text{ where } \Theta(x) \text{ is some operator}$$

* Suppose that we are dealing with bosons: $\psi(x) = \psi_B(x)$, hence

$$[\psi_B(x), \psi_B^\dagger(x')] = \delta(x-x')$$

Hence: $$e^{i\Theta(x)}\rho(x)^{1/2}\rho(x')^{1/2}e^{-i\Theta(x')} - \rho(x')^{1/2}e^{-i\Theta(x')}e^{i\Theta(x)}\rho(x)^{1/2} = \delta(x-x')$$

If we assume that $[\Theta(x), \Theta(x')] = 0$ and that for $x \neq x'$

$\rho(x)^{1/2}e^{-i\Theta(x')} = e^{-i\Theta(x')}\rho(x)^{1/2}$, then:

$$[\psi_B(x), \psi_B^\dagger(x')] \underset{x\neq x'}{=} e^{i\theta(x)}\rho(x)^{1/2}\rho(x')^{1/2}e^{-i\theta(x')} - \underbrace{\underbrace{\rho(x')^{1/2}e^{i\theta(x)}}_{e^{i\theta(x)}\rho(x')^{1/2}}\underbrace{e^{-i\theta(x')}\rho(x)}_{\rho(x)^{1/2}e^{-i\theta(x')}}}_{\rho(x)^{1/2}\rho(x')^{1/2}} = 0$$

For $x = x'$:

$$[\psi_B(x), \psi_B^\dagger(x)] = e^{i\theta(x)}\rho(x)e^{-i\theta(x)} - \rho(x) = 1$$

Hence: $\rho(x)e^{-i\theta(x)} - e^{-i\theta(x)}\rho(x) = e^{-i\theta(x)}$

. Hence a sufficient condition to satisfy $[\psi_B(x), \psi_B^\dagger(x')] = \delta(x-x')$

is: $[\rho(x), e^{-i\theta(x')}] = \delta(x-x')e^{-i\theta(x')}$

. If the density is only the $p=0$ part then: $\rho(x) \simeq \rho_0 - \frac{\nabla\phi(x)}{\pi}$

and hence:

$$[-\tfrac{1}{\pi}\nabla\phi(x), e^{-i\theta(x')}] = \frac{i}{\pi}[\nabla\phi, \theta(x')]e^{-i\theta(x')} = \delta(x-x')e^{-i\theta(x')}$$

$$\Rightarrow [\tfrac{1}{\pi}\nabla\phi(x), \theta(x')] = -i\delta(x-x')$$

$$\Rightarrow \partial_x[\tfrac{\phi(x)}{\pi}, \theta(x')] = -i\delta(x-x') \Longrightarrow \quad \partial_x \operatorname{sign}(x-x') = 2\delta(x-x')$$

$$\Rightarrow [\phi(x), \theta(x')] = i\frac{\pi}{2}\operatorname{sign}(x'-x)$$

This is exactly the result we obtained in p. 13.

* If we consider also higher harmonics ($p\neq 0$)

$$[\rho(x), e^{-i\theta(x')}] = [\rho_0 - \tfrac{1}{\pi}\nabla\phi(x), e^{-i\theta(x')}]\sum_p e^{i2p(\pi\rho_0 x - \phi(x))}$$

$$+ (\rho_0 - \tfrac{1}{\pi}\nabla\phi(x))\sum_p e^{i2p\pi\rho_0(x)}\underbrace{[e^{-2ip\phi(x)}, e^{-i\theta(x')}]}$$

$$\hookrightarrow e^{-2ip\phi(x)}e^{-i\theta(x')} - e^{-i\theta(x')}e^{-2ip\phi(x)} = \quad \text{Baker-Hausdorff formula } e^Be^A = e^Ae^Be^{-[A,B]}$$

$$= e^{-2ip\phi(x)}e^{-i\theta(x')}\{1 - e^{2p[\phi(x),\theta(x')]}\} =$$

$$= e^{-2ip\phi(x)}e^{-i\theta(x')}[1 - e^{i\pi p\operatorname{sign}(x'-x)}]$$ which is zero when $x = x'$.

* In the continuum limit only $p=0$ survives and we will assume that ϕ and θ must fulfill the commutation

$$\left[\frac{1}{\pi}\nabla\phi(x), \theta(x')\right] = -i\delta(x-x')$$

$$\Downarrow$$

when $x=x'$: $\frac{1}{\pi}\nabla\phi(x)\theta(x) - \frac{1}{\pi}\theta(x)\nabla\phi(x)$

$$= -\frac{1}{\pi}\phi(x)\nabla\theta(x) + \frac{1}{\pi}\nabla\theta(x)\phi(x) = \left[\phi(x), \frac{\nabla\theta(x)}{\pi}\right]$$

Hence $\left[\phi(x), \frac{1}{\pi}\nabla\theta(x')\right] = i\delta(x-x')$

Consequently (as we know from p. 13)

$$\Pi(x) = \frac{1}{\pi}\nabla\theta(x)$$

is the <u>canonically conjugate momentum</u> to $\phi(x)$.

* Let's come back to the single-particle operator. Recall that $\psi_B^\dagger(x) = \rho(x)^{1/2} e^{-i\theta(x)}$. We may substitute the expression of $\rho(x)$ to get $\psi_B^\dagger$ as a function of ϕ and θ:

$$\psi_B^\dagger(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]^{1/2} \sum_p e^{i2p(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)}$$

$$\rho(x) = \nabla\phi_e(x)\sum_n \delta(\phi_e(x) - 2\pi n) \rightarrow \rho(x)^{1/2} \sim \nabla\phi_e(x)^{1/2}\sum_n \delta(\phi_e(x) - 2\pi n)$$

(<u>Note</u>: Recall that $\rho(x) = \nabla\phi_e(x)\sum_n \delta(\phi_e(x)-2\pi n)$ up to a normalization factor that depends on cut-off at short distances. Recall the factor $1/\sqrt{2\pi a}$ in e.g. p. 14).

* This is for the bosonic case. Let's see now for the fermionic case. Now one has to satisfy anticommutation $[\psi_F(x), \psi_F^\dagger(x')]_+ = \delta(x-x')$. This may be done by adding something to the previous expression of $\psi_B^\dagger(x)$ such that one gets a minus sign when two fermions are commuted (this is an example of a so-called <u>Wigner-Jordan transformation</u>).
Actually it's enough to add:

$$\psi_F^\dagger(x) = \psi_B^\dagger(x)\, e^{i\phi_e(x)/2} = \left[\rho_0 - \frac{\nabla\phi(x)}{\pi}\right]^{1/2} \sum_p e^{i(2p+1)(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)}$$

* We can compare the expression of $\psi_F^\dagger(x)$ to that obtained in p. 14 for the Tomonaga-Luttinger model.

For spin-less fermions $\rho_0 = k_F/\pi$.

Recall from p. 28 that for spin-less fermions we had:

$$\rho(x) = -\frac{1}{\pi}\nabla\phi(x) + \frac{1}{2\pi\alpha}\left[e^{2ik_Fx-2i\phi(x)} + e^{-2ik_Fx}e^{2i\phi(x)}\right]$$

Let's compare this with the expression of p. 71.
Very clearly, the two structures are very similar. However, in the Tomonaga-Luttinger model we had only the $p=0$ and $p=\pm 2k_F$.
Higher harmonics were absent due to an artefact of the strictly linear dispersion (p. 5).

* Let's have a look to the Hamiltonian, and how is it written in terms of θ and ϕ.

. First of all the term $(\nabla\phi(x))^2$ must be there. These terms come from e.g. interaction terms $\sim \int dx\, \rho(x)^2$ (and the $p=0$ term is hence $(\nabla\phi)^2$).

. Also $(\nabla\theta(x))^2$ must be there. For example the part of $\psi_B^\dagger(x)$ containing less powers of $(\nabla\phi)$ (and hence the most relevant part in terms of decaying of correlations) is:

$$\psi^\dagger(x) = \rho_0^{1/2} e^{-i\theta(x)} \longrightarrow (\nabla\psi^\dagger(x))(\nabla\psi(x)) = \frac{\rho_0}{2m}(\nabla\theta(x))^2$$

and hence the kinetic energy $H_K = \int \frac{dx}{2m}(\nabla\psi^\dagger(x))(\nabla\psi(x))$ depends on $(\nabla\theta(x))^2$.

. Of course the prefactors of $(\nabla\phi)^2$ and $(\nabla\theta)^2$ can't be obtained reliably (they depend, as we already showed, from interactions) but the important point is that the Hamiltonian depends on $(\nabla\phi)^2$ and $(\nabla\theta)^2$.

. The Hamiltonian cannot depend on $\nabla\phi\,\nabla\theta$. This is because the system has mirror symmetry $x \to -x$, i.e. $\rho(x) \longleftrightarrow \rho(-x)$, $\psi(x) \longleftrightarrow \psi(-x)$

This means that $\phi(-x) = \phi(x)$; $\Theta(-x) = \Theta(x)$

$$\nabla\phi(-x) = \nabla\phi(x), \quad \nabla\Theta(-x) = -\nabla\Theta(x)$$

Hence $\nabla\phi(-x)\,\nabla\Theta(-x) = -\nabla\phi(x)\nabla\Theta(x)$ and hence doesn't show mirror symmetry and can't be part of the Hamiltonian.

. Hence the most general Hamiltonian describing the <u>low-energy properties of a massless 1D system</u> is:

$$H = \frac{\hbar}{2\pi}\int dx \left[\frac{uK}{\hbar^2}(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\phi(x))^2\right]$$

i.e. of the form we already got from the Tomonaga-Luttinger liquid for spinless fermions.

(<u>Note</u>: recall that <u>massive</u> cases cannot be reduced in the RG sense to the previous form due to the RG-relevant character of the cosine terms)

. The two coefficients <u>u and K</u> totally characterise the low-energy properties of any massless 1D system. It may be difficult to characterize them but once they are fixed, all properties of the system are determined.

* This point is <u>extremely important</u>.

. All bosonization formulas we got for the Tomonaga-Luttinger model are actually generic (the only artefact was the absence of harmonics).

This includes e.g. correlation functions and phase diagrams.

. The bosonic representation and the quadratic Hamiltonian play in 1D the role of the Fermi liquid theory in higher dimensions.

This is the so-called <u>Luttinger liquid theory</u>.

It just depends on two parameters: u and K. Once given we may obtain all asymptotic properties of correlation functions both for bosons and for fermions!

- For example, it's easy to see using the expression for $\rho(x)$ in p. 71 and similar calculations as in p. 28 that

$$\langle \rho(\vec{r})\rho(0)\rangle = \rho_0^2 + \frac{K}{2\pi^2}\frac{y_\alpha^2 - x^2}{(x^2+y_\alpha^2)^2} + \underbrace{\rho_0^2 A_2 \cos(2\pi\rho_0 x)\left(\frac{\alpha}{r}\right)^{2K} + \rho_0^2 A_4 \cos(4\pi\rho_0 x)\left(\frac{\alpha}{r}\right)^{8K}}_{\substack{\text{These extra terms are not in p. 29}\\ \text{and result from the extra harmonics in p. 71}\\ \text{compared to the Tomonaga-Luttinger model.}}} + \dots$$

- The fact that all asymptotic properties decay with exponents given by a single parameter (the Luttinger parameter K) is very remarkable and universal.
- One may proceed in a similar fashion for spinful fermions. As we did already, one may bosonize each species, introducing charge and spin $\phi_{\rho,\sigma} = (\phi_\uparrow \pm \phi_\downarrow)/\sqrt{2}$, and proceed similarly as above. However with spinful fermions one must be careful with the $g_{1\perp}$ terms (i.e. the cosine part in the spin Hamiltonian which departs from the quadratic form). This makes the calculation of correlation functions a little bit more tricky.

In particular, the density-density correlation function for a Luttinger liquid with spin is:

$$\langle \rho(r)\rho(0)\rangle = \rho_0^2 + \frac{K}{\pi^2}\frac{y_\alpha^2 - x^2}{(x^2+y_\alpha^2)^2} + \rho_0^2 A_2 \cos(2\pi\rho_0 x)\left(\frac{\alpha}{r}\right)^{K_\rho + K_\sigma} + \underbrace{\rho_0^2 A_4 \cos(4\pi\rho_0 x)\left(\frac{\alpha}{r}\right)^{4K_\rho}}_{} + \dots$$

Note that the $4k_F$ component does not depend on the spin part.
For spin-isotropic interactions ($K_\sigma = 1$) the $4k_F$ part dominates for

$$4K_\rho < K_\rho + K_\sigma = K_\rho + 1 \longrightarrow \boxed{K_\rho < 1/3}$$

- Addendum: Brief comment on operator product expansion.
- For the case with spin 1/2 we define

$$\rho_{\uparrow,\downarrow}(x) = \left[\rho_{0\uparrow\downarrow} - \frac{1}{\pi}\nabla\phi_{\uparrow\downarrow}\right]\sum_p e^{i2p(\pi\rho_{0\uparrow\downarrow}x - \phi_{\uparrow\downarrow}(x))}$$

$$\rho_{TOT}(x) = \rho_\uparrow(x) + \rho_\downarrow(x) = \left[\rho_{0\uparrow} + \rho_{0\downarrow} - \frac{1}{\pi}\nabla(\phi_\uparrow + \phi_\downarrow)\right]$$

$$+ \left[\rho_{0\uparrow} - \frac{1}{\pi}\nabla\phi_\uparrow\right]\left[e^{i2(\pi\rho_{0\uparrow}x - \phi_\uparrow(x))} + h.c.\right] + \left[\rho_{0\downarrow} - \frac{1}{\pi}\nabla\phi_\downarrow\right]\left[e^{i2(\pi\rho_{0\downarrow}x - \phi_\downarrow(x))} + h.c.\right]$$

$$+ \left[\rho_{0\uparrow} - \frac{1}{\pi}\nabla\phi_\uparrow\right]\left[e^{i4(\pi\rho_{0\uparrow}x - \phi_\uparrow(x))} + h.c.\right] + \left[\rho_{0\downarrow} - \frac{1}{\pi}\nabla\phi_\downarrow\right]\left[e^{i4(\pi\rho_{0\downarrow}x - \phi_\downarrow(x))} + h.c.\right]$$

let $\rho_{0\uparrow} = \rho_{0\downarrow} = \rho_0/2 = k_F/\pi$ then (using $\phi_{\rho,\sigma} = \frac{1}{\sqrt{2}}(\phi_\uparrow \pm \phi_\downarrow)$)

$$\rho_{TOT}(x) = \rho_0 - \frac{\sqrt{2}}{\pi}\nabla\phi_\rho(x) + \rho_0\left[e^{i(2k_Fx - \sqrt{2}\phi_\rho(x))}\cos\sqrt{2}\phi_\sigma + h.c.\right]$$

$$+ \rho_0\left[e^{i(4k_Fx - 2\sqrt{2}\phi_\rho(x))}\cos\sqrt{8}\phi_\sigma + h.c.\right] + \dots$$

(recall that terms $\nabla\phi\, e^{i\phi}$ don't contribute in correlation functions)

(recall the sine-Gordon contribution $(g_{1\perp} = 0)$)

- If we forget the sine-Gordon contribution

$$\langle\rho_{TOT}(r)\rho_{TOT}(0)\rangle_0 \cong \rho_0^2 + \frac{2}{\pi}\langle\nabla\phi_\rho(x)\nabla\phi_\rho(0)\rangle$$

$$+ \rho_0^2\left[e^{i2k_Fx}\langle e^{-i\sqrt{2}(\phi_\rho(x) - \phi_\rho(0))}\rangle + e^{-2ik_Fx}\langle e^{i\sqrt{2}(\phi_\rho(x) - \phi_\rho(0))}\rangle\right] \times$$

$$\times 2\left[\langle e^{i\sqrt{2}(\phi_\sigma(x) - \phi_\sigma(0))}\rangle + \langle e^{-i\sqrt{2}(\phi_\sigma(x) - \phi_\sigma(0))}\rangle\right]$$

$$+ \rho_0^2\left[e^{i4k_Fx}\langle e^{-i2\sqrt{2}(\phi_\rho(x) - \phi_\rho(0))}\rangle + h.c.\right] \times 2\left[\langle e^{2i\sqrt{2}(\phi_\sigma(x) - \phi_\sigma(0))}\rangle + h.c.\right]$$

$$= \rho_0^2 + \frac{K_\rho}{2\pi^2}\frac{y_\alpha^2 - x^2}{(x^2 + y_\alpha^2)^2} + 2\rho_0^2\cos(2k_Fx)\left(\frac{\alpha}{r}\right)^{(K_\rho + K_\sigma)}$$

$$+ 2\rho_0^2\cos(4k_Fx)\left(\frac{\alpha}{r}\right)^{4(K_\rho + K_\sigma)} + \dots$$

- However, if $g_{1\perp} \neq 0$ we cannot forget the sine-Gordon contribution and this leads to surprises. Let's see that.

* Recall that (p. 46) the action may be written as:

$$S = S_0 + \frac{2g}{(2\pi\alpha)^2 u}\int d^2r \cos\sqrt{8}\phi_\sigma(r)$$

We may hence expand in g:

$$\langle O\rangle = \frac{1}{Z}\int D\phi\, e^{-S_0}\, O\left[1 - \int d^2r \frac{2g}{(2\pi\alpha)^2 u}\cos\sqrt{8}\phi_\sigma + \frac{1}{2}\left[\int d^2r \frac{2g}{(2\pi\alpha)^2 u}\cos\sqrt{8}\phi_\sigma(r)\right]^2\right]$$

with $Z \cong \int D\phi\, e^{-S_0}\left[1 - \int d^2r \frac{2g}{(2\pi\alpha)^2 u}\cos\sqrt{8}\phi_\sigma + \frac{1}{2}\left[\int d^2r\left(\frac{2g}{(2\pi\alpha)^2 u}\right)\cos\sqrt{8}\phi_\sigma(r)\right]^2\right]$

Let $\langle O\rangle_0 = \frac{1}{Z_0}\int D\phi\, e^{-S_0}\, O$ with $Z_0 = \int D\phi\, e^{-S_0}$

Then (note that terms with ~~1~~ odd number of cosines vanish.

$$\langle \rho(r)\rho(0)\rangle \cong \langle \rho(r)\rho(0)\rangle_0 + \frac{1}{2}\langle \rho(r)\rho(0)\left[\int d^2r'\left(\frac{2g}{(2\pi\alpha)^2 u}\right)\cos\sqrt{8}\phi(r')\right]^2\rangle_0$$

In the extra term we have that

$$\rho(r) \longrightarrow \tilde{\rho}(r) = \rho(r)\int d^2r'\left(\frac{2g}{(2\pi\alpha)^2 u}\cos\sqrt{8}\phi(r')\right)$$

and the last term is hence of the form $\langle\tilde{\rho}(r)\tilde{\rho}(0)\rangle$

. If r and r' are separated this extra term leads after averaging to an additional power law decay which is less relevant than the $\cos\sqrt{8}\phi$ terms coming in $\langle\rho(r)\rho(0)\rangle_0$. However if r and r' are close within a cut-off distance (α) then $\tilde{\rho}(r)$ contains terms of the form (only these terms are relevant for $\langle\tilde{\rho}(r)\tilde{\rho}(0)\rangle_0$)

$$\cos\sqrt{8}\phi_\sigma(r)\int d^2r' \frac{2g}{(2\pi\alpha)^2 u}\cos\sqrt{8}\phi_\sigma(r')$$

$$\simeq \cos^2\sqrt{8}\phi_\sigma(r)\cdot\frac{2g}{(2\pi\alpha)^2 u}\alpha^2 = \frac{g}{4\pi^2 u}\left[1+\cos 2\sqrt{8}\phi_\sigma(r)\right]$$

The term with $2\sqrt{8}\phi_\sigma$ decays very fast but we get a constant term $g/4\pi^2 u$ which is hence a constant!

- Hence (we only write the most relevant term)

$$\langle \tilde{\rho}(r)\,\tilde{\rho}(0)\rangle_0 \simeq \frac{1}{2}\left(\frac{g}{4\pi^2 u}\right)^2 \cdot 4 \left\langle e^{i4k_F x} e^{-i\sqrt{8}(\phi_\rho(x)-\phi_\rho(0))} + h.c. \right\rangle$$

$$= \cancel{4}\left(\frac{g^2}{4\pi^2 u}\right)^2 \cos 4k_F x \left(\frac{\alpha}{r}\right)^{4K_\rho}$$

- Hence the density-density correlation function goes actually is:

$$\langle \rho(r)\rho(0)\rangle \sim \rho_0^2 A_2 \cos(2\cancel{k_F}x)\left(\frac{\alpha}{r}\right)^{(K_\rho+K_\sigma)} + \rho_0^2 A_4 \cos(4k_F x)\left(\frac{\alpha}{r}\right)^{4K_\rho}$$

Note: This tells us that in p. 76 we could have written a better expression for the density where the $4k_F$ term loses the $\cos\sqrt{8}\phi_\sigma$ part: $\rho_0 e^{i[4k_F x - 2\sqrt{2}\phi_\rho]} + h.c.$

where the constants $A_{2,4}$ are non-universal objects.

- Note that compared to the naive non-sine-Gordon result, now the $4k_F$ terms doesn't decay as $(\alpha/r)^{4(K_\rho+K_\sigma)}$ but as $\left(\frac{\alpha}{r}\right)^{4K_\rho}$.

This is a huge difference.

- If the $4k_F$ term decays as $\left(\frac{\alpha}{r}\right)^{4(K_\rho+K_\sigma)}$ then it decays always faster as $\left(\frac{\alpha}{r}\right)^{(K_\rho+K_\sigma)}$ and hence the $2k_F$ term always dominate. But if $\left(\frac{\alpha}{r}\right)^{4K_\rho}$ then if $4K_\rho < (K_\rho+K_\sigma)$ the $4k_F$ term dominates.

i.e. if $K_\rho < K_\sigma/3$, then the $4k_F$ term dominates.

- For spin-isotropic interaction $K_\sigma = 1$ and the condition reduces to $\boxed{K_\rho < 1/3}$

- This example shows that one must be extremely careful when calculating correlation functions, since the contraction with other operators can lead to a part that decays more slowly than each of the original operators. This is the so-called <u>operator product expansion</u>.