• REFINEMENTS

* In the following lectures we will have a look to some additional complications that may appear. These include long-range interactions, the effects of a discrete lattice (Mott transition), the effects of an external magnetic field (i.e. when spin rotation invariance is violated), and the treatment of marginally relevant (in the RG sense) contributions.

* LONG-RANGE INTERACTIONS

Up to now we have just considered short range interactions. Let's consider now an interaction $V(x)$ with an arbitrary spatial dependence. The corresponding interaction Hamiltonian is:

[Note: $\rho_{TOT} = \rho_\uparrow + \rho_\downarrow$. Note the factor $\sqrt{2}$ compared to p. (35)]

$$H_{INT} = \frac{1}{2}\int dx \int dx'\, V(x-x')\,\rho_{TOT}(x)\,\rho_{TOT}(x') = \frac{1}{2\Omega}\sum_q V(q)\,\rho_{TOT}(q)\,\rho_{TOT}(-q)$$

Recall from our discussion of the Tomonaga-Luttinger model that (p. (35))

$$\rho(x) = \frac{1}{\sqrt{2}}(\rho_\uparrow + \rho_\downarrow) = \frac{1}{\sqrt{2}}\left[\rho_{R\uparrow} + \rho_{L\uparrow} + \rho_{R\downarrow} + \rho_{L\downarrow} + \psi^\dagger_{R\uparrow}\psi_{L\uparrow} + \psi^\dagger_{L\uparrow}\psi_{R\uparrow} + \psi^\dagger_{R\downarrow}\psi_{L\downarrow} + \psi^\dagger_{L\downarrow}\psi_{R\downarrow}\right]$$

$$= \frac{1}{\sqrt{2}}\left\{-\frac{1}{\pi}\nabla(\phi_\uparrow + \phi_\downarrow) + \frac{1}{2\pi\alpha}\left[e^{2ik_Fx}e^{i\sqrt{2}(\phi_\rho+\phi_\sigma)} + e^{2ik_Fx}e^{i\sqrt{2}(\phi_\rho-\phi_\sigma)} + h.c.\right]\right\}$$

$$= \underbrace{-\frac{1}{\pi}\nabla\phi_\rho}_{q\sim 0 \text{ part}} + \underbrace{\frac{\sqrt{2}}{\pi\alpha}\cos(2k_Fx + \sqrt{2}\phi_\rho)\cos(\sqrt{2}\phi_\sigma)}_{q\sim 2k_F \text{ part}}$$

Since we are mostly interested in the long-range part of the interaction we will only keep the $q\sim 0$ part. Note that the $q\sim 0$ part just involves the charge part (via $\nabla\phi_\rho$) (Note: the $2k_F$ part does depend on ϕ_σ).

$$\text{Hence}\quad \rho(x) \simeq -\frac{1}{\pi}\nabla\phi_\rho$$

$$\phi_\rho(x) = \frac{1}{\Omega}\sum_q \phi_\rho(q)e^{iqx} \longrightarrow \nabla\phi_\rho = \frac{1}{\Omega}\sum_q (iq)\phi_\rho(q)e^{iqx}$$

$$\longrightarrow \rho(x) \simeq \frac{1}{\Omega}\sum_q\left(\frac{-iq}{\pi}\phi_\rho(q)\right)e^{iqx} \longrightarrow \rho(q) \simeq \frac{-iq}{\pi}\phi_\rho(q)$$

Then: $$H_{INT} = \frac{1}{\cancel{2}\Omega}\sum_q \frac{q^2}{\pi^2}V(q)\,\phi_\rho(q)^*\,\phi_\rho(q)$$

(Recall that $\phi_\rho(-q) = \phi_\rho(q)^*$)

(Note: the factor 2 in the denominator disappears once $\rho_{TOT} = \sqrt{2}\rho$ with ρ in p. (35)).

* If we add this to the kinetic energy (we just write the charge sector)

$$H_0 = \frac{1}{2\pi}\int dx\, v_F\left[(\nabla\theta_\rho)^2 + (\nabla\phi_\rho)^2\right]$$

$$= \frac{1}{2\pi}\,\frac{1}{\Omega}\sum_q v_F q^2\left[\theta_\rho^*(q)\,\theta_\rho(q) + \phi_\rho^*(q)\,\phi_\rho(x)\right]$$

we obtain the Hamiltonian:

$$\tilde{H} = \sum_\rho \frac{u_\rho(q)}{2\pi\Omega}q^2\left[K_\rho(q)\,\theta_\rho^*(q)\,\theta_\rho(q) + \frac{1}{K_\rho(q)}\phi_\rho(q)^*\phi_\rho(q)\right]$$

where $u_\rho(q)\,K_\rho(q) = v_F$

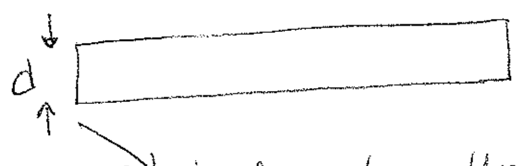
$$\frac{u_\rho(q)}{K_\rho(q)} = v_F + \frac{2V(q)}{\pi}$$

* We recover hence once more a quadratic Hamiltonian. We may then evaluate the correlation functions using our discussion of p. ⑳-㉗. The main difference is that we should keep the q-dependence of $u_\rho(q)$ and $K_\rho(q)$ in the integrals over q (this gives a different functional form for $F_1(\vec{r})$ and $F_2(\vec{r})$ in pages ㉓ and ㉔ as those of p. ㉕).

- Two cases are possible:

- $V(q\to 0)$ remains finite. Then the asymptotic form of the correlation functions is the same as usual for a quadratic Hamiltonian but with the Luttinger parameters $K_\rho(0)$, $u_\rho(0)$ [i.e. it's the same as having a δ potential with strength $V(q=0)$. In other words all systems with short-range interactions (i.e. with finite $\int dx\, V(x)$) are Luttinger liquids, i.e. they present power law correlation functions (in the massless sector).

- $V(q\to 0)$ diverges → the q-dependence of the interactions is crucial and must be kept explicitely. This is the case of an unscreened Coulomb interaction in quantum wires (1D electrons).



The wire has a width d. At distances larger than d we can replace the system by a purely 1D system.

↘ d plays here the role of a short-distance cut-off.

- For $r \gg d \rightarrow V(r) \approx \frac{e^2}{4\pi\varepsilon |r|}$

but for $r \sim d$ the electrons "feel" the other dimension.
We can take this phenomenologically into account by considering

$$V(r) \simeq \frac{e^2}{k\sqrt{r^2+d^2}} \qquad (k = 4\pi\varepsilon)$$

which has a simple Fourier transform in terms of Bessel function:

$$V(q) = \frac{2e^2}{k} K_0(|q|d)$$

- Although the $q \rightarrow \infty$ behaviour depends on d (obviously since d is the small spatial scale, and hence affects the large momenta) the $q \rightarrow 0$ behaviour is of course cut-off independent. In particular

$$V(q \rightarrow 0) \simeq 2\int_d^{1/q} \frac{dx\, e^2}{k|x|} \simeq \log(1/qd) \cdot \frac{2e^2}{k}$$

- Hence

$$u_\rho(q)^2 \simeq v_F \left[v_F + \frac{2V(q)}{\pi} \right] \overset{q \rightarrow 0}{\simeq} v_F \frac{2V(q)}{\pi} \simeq \frac{4 v_F e^2}{\pi k} \ln(1/qd)$$

- Recall that the (linear) excitation spectrum (p. ㊵) is given by the Luttinger speed. Hence for the charge mode the spectrum is

$$\boxed{\omega(q) = u_\rho(q)|q| \simeq \left[\frac{4 v_F e^2}{\pi k} \right]^{1/2} |q| \ln\left(\frac{1}{qd}\right)}$$

i.e. the spectrum is not any more linear in q due to the logarithmic divergence of $V(q \rightarrow 0)$.
(Note: that the spin sector remains linear since it isn't affected by the long-range interaction).

- Hence the long-range interactions change the nature of the spectrum of excitations which are no more phonon-like (i.e. sound waves) (it is a so-called plasmon dispersion)

* Correlations are also very affected by long-range interactions. That's easy to see because

$$K_\rho(q)^2 = \frac{v_F}{v_F + 2V(q)/\pi} \simeq \frac{\pi v_F \hbar}{4e^2}\,\frac{1}{\ln(1/qd)}$$

Hence $K_\rho(q) \xrightarrow[q\to 0]{} 0$ → the effective Luttinger parameter decay to zero. As a consequence one expects that the correlations involving ϕ_ρ will decay more slowly than any power law.

· Hence, the charge part is nearly ordered (obviously not completely because we are in 1D but much more ordered than a Luttinger liquid) (Note: of course the correlations involving θ_ρ decay very fast now because they go with $1/K_\rho$. As always ϕ_ρ and θ_ρ are conjugate).

· As a result

$$\langle(\phi_\rho(x,0)-\phi_\rho(0,0))^2\rangle \sim \sqrt{\ln x}$$

and the $2k_F$ and $4k_F$ parts of the density-density correlation (for $K_\sigma = 1$) are (p. (76))

$$\langle\rho(x)\rho(0)\rangle = \dots + A_1' \cos(2k_F x)\,\frac{1}{x}\,e^{-c\sqrt{\ln x}} + A_2' \cos(4k_F x)\,e^{-4c\sqrt{\ln x}} + \dots$$

where c is a constant, as well as A_1' and A_2'.

· It's clear that the $4k_F$ density-wave dominates now. (recall from p. (76) that $4k_F$ dominates for $K_\rho < 1/3$ which is the case now since $K_\rho(q\to 0) \to 0$). Actually the $4k_F$ part decays as $\boxed{e^{-4c\sqrt{\ln x}}}$, i.e. extremely slowly. The charge is hence as ordered as it may be in 1D. A $4k_F$ modulation has the periodicity of the mean distance between the electrons. One has hence a very strong tendency to crystallize (1D equivalent of a Wigner crystal) → in 1D it occurs regardless of the density due to long-range interactions (contrary to higher-dimensions where one has Lindemann criterion). However, in 1D perfect crystallization is of course precluded.

× MOTT TRANSITION

Up to now we have not considered any underlying lattice. At this point we will discuss in some detail one of the most important consequences of the existence of a lattice, namely the Mott insulator.

• Mott-insulator

. let's consider a spin-$1/2$ system in a lattice (we reduce to the lowest band).

. let's consider the case of 1 particle per site (this is the so called half filling; of course if there're 2 particles per site they must be $\uparrow$ and $\downarrow$, and in that case we have a band insulator, i.e. the band is full)

. let's consider that there's a repulsion U for particles on site and that the kinetic energy (i.e. the hopping) is t.

. If $U \gg t \Rightarrow$ the plane wave state (hence a delocalized state) is not very favourable, and repulsion favors the localization of particles at each site. This is the idea of Mott insulator.

. Away from half-filling, if the system is weakly doped with holes (i.e. we have a lower filling than 1) the holes can propagate without any repulsion, and the system is metallic, but with a number of carriers given by the doping.

. There's hence a transition between Mott-insulator and metal, which may occur due to two reasons. Either because we change the ratio U/t or because we go away of half-filling changing the doping. We will see later on that these two ways of crossing the transition are not equivalent.

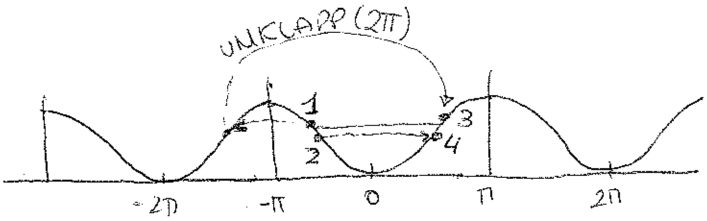
• Umklapp processes

. The existence of a lattice means that the wavevector is in fact defined modulo a vector Q of the reciprocal lattice. In 1D this means that it's defined up to a multiple of $\frac{2\pi}{a} \equiv Q$, with a the lattice spacing.

Then, in an interaction processes we may have

- $k_1 + k_2 = k_3 + k_4$ → processes that truly conserve momentum
- $k_1 + k_2 - k_3 - k_4 = Q$ → these are the so called Umklapp processes

Since we are interested in the 1D physics at low energies, we should ask ourselves whether the umklapp processes are relevant at the Fermi surface. In 1D these processes are:



Hence $k_1 + k_2 - k_3 - k_4 = 2\pi$

(we take $a = 1$)

Since in the graph $k_1 = k_2 \doteq k_F$
$k_3 = k_4 = k_F$

the umklapp process is just possible if

$$4k_F = 2\pi$$

Recall that $k_F = \pi$ means full-filling. Hence $k_F = \pi/2$ means half-filling.

Hence umklapp processes just exist at half filling.

- Umklapp processes in the bosonization formalism: sine-Gordon in the charge sector

let's consider the interaction

$$H = \int dx \, U \rho_\uparrow(x) \rho_\downarrow(x)$$

Recall that in our discussion of the continuum limit (i.e. without lattice) we considered only processes that conserved momentum (p.⑮). This was the case of processes g_4 (which occurred at the same side of the Fermi-surface), g_2 (which did not change sides) and g_1 (which just swapped sides). We did not consider processes going with $4k_F$. Recall that (p.⑮)

Thus processes are however there.

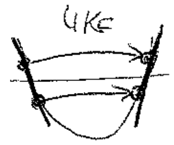
$$\rho(x) = \underbrace{\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L}_{\substack{q \sim 0 \\ \text{part}}} + \underbrace{\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R}_{\substack{q \sim \pm 2k_F \\ \text{part}}}$$

The point is that the terms of $\rho(x)\rho(x)$ going with $q \neq 0$ oscillate fast and are eliminated when integrating, which of course is another way of saying that the momentum must be conserved.

· In the presence of the lattice umklapp processes may be relevant (they are at half-filling).

The $4k_F$ processes contribute with terms of the form

$$\psi^\dagger_{R\uparrow}\psi_{L\uparrow}\psi^\dagger_{R\downarrow}\psi_{L\downarrow} + h.c.$$



Using the expressions of p. (35)

$$\psi^\dagger_{R\uparrow}\psi_{L\uparrow}\psi^\dagger_{R\downarrow}\psi_{L\downarrow} = \frac{1}{(2\pi\alpha)^2} e^{-4ik_Fx} e^{i2\sqrt{2}\phi_\rho}$$

and hence the ~~umklapp~~ $4k_F$ processes lead to the Hamiltonian

$$H = \frac{U}{(2\pi\alpha)^2}\int dx\, [e^{-i4k_Fx} e^{i2\sqrt{2}\phi_\rho(x)} + h.c.]$$

· At half filling $4F = 2\pi/a$, since in a lattice $x = na$, then $e^{-i4k_Fx} = 1$ → These terms do not oscillate → they are of course the umklapp processes discussed before.

· Hence, in a lattice with one fermion per site the charge part of the Hamiltonian becomes

$$H = H_0 + \underbrace{\frac{2g_3}{(2\pi\alpha)^2}\int dx \cos\sqrt{8}\phi_\rho(x)}_{\text{Umklapp}}$$

[g_3 depends on the microscopic model used

while the spin part is unchanged. We have hence a sine-Gordon form in the charge sector. This is the same type of Hamiltonian we found for the spin sector (p. (36)). Recall that this term leads to a phase transition from massless to massive phases (depending now on K_ρ and g_3).

· In fact, the fact that at half filling the charge sector gets similar properties as those of the spin sector is of course not by chance. A particle-hole transformation in one of the species allows to pass from spin to charge sector and viceversa. This particle-hole symmetry explains the similarities between both sectors.

- Away from half-filling

- Let's see what happens if we move away from half-filling.

This may be done in 2 ways:

(i) We introduce a chemical potential term (recall p.㊳)

$$H \longrightarrow H - \mu\int dx\,(\rho_\uparrow(x)+\rho_\downarrow(x)) = H + \mu\frac{\sqrt{2}}{\pi}\int dx\,(\nabla\phi_\rho)$$

(ii) We may work with a fixed number of particles but introduce a doping d ($d=1$ for a filled band and $d=0$ for a half-filled one).

then $4k_F = \frac{2\pi}{a}+\delta$ with $\delta = 2\pi d$.

Then the umklapp terms go as $e^{i4k_Fx} = e^{i\delta x}$. If δ is small then we should keep the δx term, and this shifts the argument of the cosine

$$H = H_0 + \frac{2g_3}{(2\pi\alpha)^2}\int dx\,\cos[\sqrt{8}\phi(x) + \delta x]$$

[Note: in p.㊳ we have seen that introducing μ amounts for shifting $\phi_\rho \to \tilde{\Phi}_\rho = \phi_\rho + \frac{\mu\sqrt{2}K_\rho}{u_\rho}x$, which hence leads to a similar shift as that introduced by the doping. However, in the presence of μ, we have $\langle\nabla\tilde{\Phi}_\rho\rangle = 0$ and not $\langle\nabla\phi_\rho\rangle$. Recall that $N = \int dx(\rho_\uparrow+\rho_\downarrow) = \frac{-\pi}{\sqrt{2}}\int dx\,\nabla\phi_\rho$, and hence $\langle\nabla\phi_\rho\rangle$ is not constrained. In the case of doping N is fixed and hence $\langle\nabla\phi_\rho\rangle$ is constrained.]

- Actually umklapps are not restricted to one particle per site, we may get umklapp for any commensurate density.

That's easy to see. Recall our discussion on phenomenological bosonization. Recall that in addition to the $2k_F$ term we have all harmonics $2pk_F$ (with non-universal amplitudes).

We may now take this formula and introduce the lattice directly in the Hamiltonian of the continuum

$$H_{lattice} = \int dx\,V(x)\,\rho(x) = V_0\int \cos(\underbrace{4k_F}_{\uparrow}x)\,\rho(x)$$

This is the lattice wavenumber we need to get one particle per site (recall p.(80))

- It's easy to see that this H_{Latt} leads to umklapp processes.

But we can easily get umklapp for all commensurate densities, such that $2pk_F = 2\pi q/a$ (with p, q integers)

Hence $e^{i2pk_Fx} \overset{x=ja}{\underset{\downarrow}{=}} e^{i2\pi\frac{q}{\cancel{a}}j\cancel{a}} = 1$ and the term survives.

- For even commensurabilities $p = 2n$, the corresponding term in the density is:

$$\sim e^{i[4nk_Fx - n\sqrt{8}\phi_\rho]}\cos[n\sqrt{8}\phi_\sigma]$$

- Note that the $\cos n\sqrt{8}\phi_\sigma(r)$ term is eliminated with the $\cos\sqrt{8}\phi_\sigma$ term in the Hamiltonian.

<u>Note</u>: as in page (76'') we may consider $\rho(r)\int d^2r'\cos\sqrt{8}\phi(r')$, which has now the term $\cos n\sqrt{8}\phi_\sigma(r)\int d^2r'\cos\sqrt{8}\phi(r')$, those r' such that $\cos\sqrt{8}\phi_\sigma(r') \simeq \cos n\sqrt{8}\phi_\sigma(r)$ lead to a constant which dominate the asymptotic behavior.]

This is of course always possible due to commensurability. close to the minimum of maximum $\cos\sqrt{8}\phi_\sigma \simeq \pm\cos n\sqrt{8}\phi_\sigma$. This isn't true without even commensurability, where the spin can't be eliminated

- We will set then the umklapp term:

$$H_{1/2n} = g_{1/2n}\int dx\cos(n\sqrt{8}\phi_\rho(x))$$

and in the presence of hopping

$$H_{1/2n} = g_{1/2n}\int dx\cos(n\sqrt{8}\phi_\rho(x) - \delta x)$$

- Note that $n=1 \longrightarrow$ half filling (o o o o o o o o)
$n=2 \longrightarrow$ quarter filling (o × o × o × o)

- For example $n=2$ umklapp $(8k_F = 2\pi)$ may be physically understood as a transfer of 4 particles from one side of the Fermi surface to the other. This demand 3 scatterings $\longrightarrow$ higher-order perturbation terms of the order of U^3/W^2 with $U \equiv$ interaction, $W \equiv$ bandwidth

- Hence, to change the properties of a commensurate system we have two ways:
  - * Mott-U: we vary the strength of interactions keeping a commensurate filling.
  - * Mott-δ: we vary the filling (or μ) keeping the interactions constant.

Both ways are not equivalent as we will see now.

- Mott-U: Luther-Emery solution

- Recall that the sine-Gordon Hamiltonian was already encountered in the spin sector of a spin-1/2 Fermi system without lattice. We solved it then using RG. Recall in particular the diagram of page (65).

- There's a phase transition (of a BKT-type) at a critical value of the Luttinger parameter K_c^*. For filling $n=1$ (half-filling) we have $K_c^* = 1$ like in p. (65). In general we will have $K_c^* = 1/n^2$

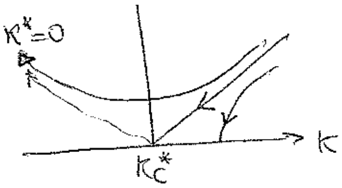
- For $K > K_c^*$ the cosine term is RG-irrelevant and the system is massless (i.e. there's no gap)

- For $K < K_c^*$ the cosine is relevant and the system is massive (i.e. a gap opens). The opening of the gap corresponds to the Mott transition.

- Note that the larger the commensurability the smaller K is needed to become a Mott insulator.
  - For $n=1 \longrightarrow K=1 \longrightarrow$ any repulsive interaction turn the half-filled system into an insulator.
  - For $n=2 \longrightarrow K=1/4 \longrightarrow$ for quarter filling we need strong and finite-range interactions

(I recall you that local interactions lead at minimum to $K_\rho = 1$) At quarter filling we need finite-range interactions obviously since the particles must be 2 sites apart from each other.

- An important property of the Mott-insulator may be understood from the RG flow (similar to that of p. (65)). At the phase transition (the separatrix) the renormalized K goes from K_c^* abruptly to zero.



Since the velocity u_ρ is not renormalized, then the compressibility $\kappa_\rho = \frac{2K_\rho}{\pi u_\rho}$ (recall p. (38)) is renormalized to zero in the massive phase. The compressibility hence drops discontinuously to zero inside the Mott-phase.

- In the Luttinger liquid phase the Hamiltonian remains quadratic up to the transition (recall that there the cosine term is irrelevant). Quadratic means Lorentz invariant, i.e. x and τ behave similarly. This is in general not always so, and to take into account that, one introduces the so called dynamical exponent$^{(z)}$, which tells us how a characteristic τ-length relates to a characteristic x-length $\xi_\tau \sim \xi_x^z$. For the Mott-U transition $z=1$. This will not be the case of a Mott-δ transition.

- To study the properties of the massive phase we will use a very useful solution found by Luther and Emery.

- Let's consider the Hamiltonian

$$H = H_\rho^0 + g_{1/2\pi} \int dx \cos[\sqrt{8}\,\phi_\rho(x)]$$

$$\underset{\text{Hamiltonian}}{\text{Quadratic}} \Rightarrow \frac{1}{2\pi}\int dx \left[uK(\nabla\theta_\rho)^2 + \frac{u}{K}(\nabla\phi_\rho)^2\right]$$

- Recall that for spin-less fermions we may use the expression of p. (14) to get

$$\psi_R^\dagger(x)\psi_L(x) = \frac{1}{2\pi\alpha} e^{2i\phi}$$

($\psi_R^\dagger\psi_L$ is a backscattering $L \curvearrowright R$)

$$\text{Hence} \quad \psi_R^\dagger(x)\psi_L(x) + h.c. = \frac{1}{\pi\alpha}\cos 2\phi$$

- We have hence recovered a cosine term but with the wrong coefficient inside the cosine.

(Note: Hence $\theta_\rho \to \tilde{\theta} = \theta_\rho / n\sqrt{2}$)

We may however re-scale ϕ_ρ: $\tilde{\phi} = n\sqrt{2}\,\phi_\rho$.

Then $n\sqrt{8}\,\phi_\rho = 2\tilde{\phi}$. The Hamiltonian then becomes:

$$\boxed{H = \frac{1}{2\pi}\int dx \left[u\tilde{K}(\nabla\tilde{\theta})^2 + \frac{u}{\tilde{K}}(\nabla\tilde{\phi})^2\right] + g_{1/2\pi}\int dx \cos(2\tilde{\phi})}$$

where $\tilde{K} = K\,2n^2$

- We can see then <u>H as the Hamiltonian of spin-less fermions.</u> The quadratic part contains the kinetic energy plus interactions between the fermions. The cosine term describes <u>backscatterings</u> of the form

$\pi\alpha g_{1/2\pi}\int dx\,(\psi_R^\dagger \psi_L + h.c.)$

- This Hamiltonian with:
  - Kinetic energy
  - Interaction
  - Backscattering

  is the so-called <u>Massive-Thirring model</u>

- The crucial point is that for $K = \frac{1}{2n^2}$ (which is deeply into the massive phase) we have $\tilde{K} = 1$, i.e. the effective spinless fermions are <u>non-interacting</u> (!!)

This is the so-called <u>Luther-Emery line</u>. On this line we have non-interacting fermions with back-scatterings:

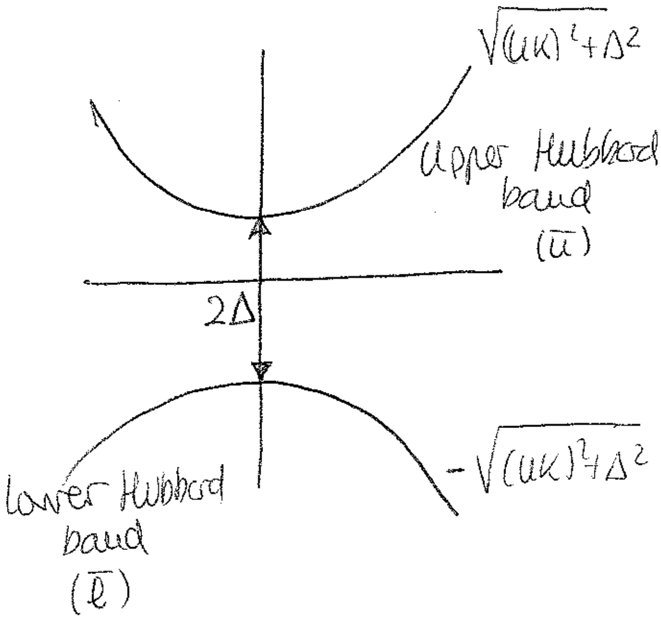
$$\boxed{H = \underbrace{\sum_k uk\,(c_{Rk}^\dagger c_{Rk} - c_{Lk}^\dagger c_{Lk})}_{\text{Tomonaga-Luttinger}} + \underbrace{\pi\alpha g_{1/2\pi}\sum_k c_{Rk}^\dagger c_{Lk} + h.c.}_{\text{Backscattering}}}$$

↳ <u>Luther-Emery Hamiltonian</u>

* This Hamiltonian is very easy to diagonalize.

Let
$$\left.\begin{aligned} C^{\dagger}_{\bar{u},k} &= \alpha_k C^{\dagger}_{Rk} + \beta_k C^{\dagger}_{Lk} \\ C^{\dagger}_{\bar{l},k} &= -\beta_k C^{\dagger}_{Rk} + \alpha_k C^{\dagger}_{Lk} \end{aligned}\right\}$$

with
$$\begin{matrix}\alpha_k \\ \beta_k\end{matrix} = \left[1 \pm \frac{uk}{2\sqrt{(uk)^2+\Delta^2}}\right]^{1/2}$$



These states have energies
$$E_{\substack{\bar{u}\\ \bar{l}}} = \pm\sqrt{(uk)^2+\Delta^2}$$

with $\Delta = \pi\alpha g_1/2\pi$.

- Hence we recover two bands: the upper and the lower Hubbard bands, which are separated by a gap given by the backscattering term.
- If one is away from the Luther-Emery line the interaction renormalises the gap Δ but the energies keep the form.
- This 2-band picture is extremely useful to understand the Mott insulator.

At commensurate filling the chemical potential is in the gap, so that all $\bar{l}$ states are filled and all $\bar{u}$ states are empty. At zero-temperature the system is obviously an insulator, actually very similar to a simple band insulator.

- However the free fermions are not the original ones but the eigenstates of the Luther-Emery Hamiltonian of p. ⑧⑧.

To understand what these excitations are we should recall that for spin-less fermion: $\rho(x) = -\nabla\phi(x)/\pi$.

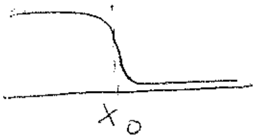
• If a particle is added at $x = x_0$ $[\rho(x) = \delta(x - x_0)]$

then: $\int_{-\infty}^{x>x_0} dx' \rho(x') = 1$

$$\parallel$$

$$\int_{-\infty}^{x>x_0} dx' \left(\frac{-\nabla\phi}{\pi}\right) = -\frac{1}{\pi}\left[\phi(x) - \phi(-\infty)\right]$$

Hence $\phi(x) = \phi(-\infty) - \pi \Longrightarrow$



A particle at x_0 manifests itself as a kink in ϕ.

• Hence the spin-less effective fermions correspond to kinks of the field Φ, and hence to kinks of ϕ_ρ. The kinks of ϕ_ρ are conversely related to density defects of the spin-1/2 Fermi system (holons). Hence the Luther-Emery excitations are simply holons.

We see that at the magic value $K = 1/2n^2$ the holons behave as free fermions. (holons include both holes and double occupancies in the more conventional Mott-insulators).

• Mott-δ : Commensurate-incommensurate transition

* Let's consider now the doped system. We will extend the RG treatment of p.(53) to treat a sine-gordon term of the form:

$$H_{1/2n} = g_{1/2n} \int dx \cos\left[n\sqrt{8}\,\phi_\rho(x) - \delta x\right]$$

The transition occurring there is the so-called commensurate-incommensurate (C-IC) phase transition, which was studied in the problem of a classical crystal of lines, where the periodicity of the crystal and that of the underlying potential are slightly mismatched.

* To illustrate the idea we confine to the $n=1$ case. We drop the ρ subindex, and take $g_{1/2n} = \frac{2g}{(2\pi\alpha)^2}$

- We may follow exactly the same steps as in p. (53), but now instead of $\cos\sqrt{8}\phi$ we have $\cos(\sqrt{8}\phi + \delta x)$. In the last equation of p. (54) this amounts for:

$$R(r_1-r_2) = e^{-a^2KF(r_1-r_2)}\left\{1+\frac{g^2}{2(2\pi\alpha)^4u^2}\int d^2R\,d^2r \sum_{\epsilon_1=\pm1} \overset{\text{This is new now}}{\boxed{e^{-i\epsilon_1\delta x}}}\, e^{-4KF(r)}\left[e^{2a\epsilon_1K[F(r_1-R-r/2)-F(r_1-R+r/2)-F(r_2-R-\frac{r}{2})-F(r_2-R+\frac{r}{2})]}-1\right]\right\}$$

- It's then easy to see that the expression of p. (55) changes into:

$$R(r_1-r_2) = e^{-a^2KF(r_1-r_2)}\left[1+\frac{2g^2}{(2\pi\alpha)^4u^2}\int d^2R\int d^2r\; e^{-4KF(r)}\cos(\delta x)\cdot\left[aK\,\bar{r}\cdot\bar{\nabla}[F(r_1-R)-F(r_2-R)]\right]^2\right]$$

- In p. (55) we had $\delta=0$, and hence there was a rotational invariance in the $(x, y=u\tau)$ plane. Here, however, the $\cos\delta x$ breaks that symmetry. As a result the x^2 and y^2 terms are not equivalent and space and time must be renormalized differently.

- Let's set:

$$F(\bar{r}-\bar{r}') = \frac{1}{2}\log\left[\frac{(x-x')^2+(y-y')^2}{\alpha^2}\right] + \frac{t}{K}\cos 2\theta_{\bar{r}-\bar{r}'}$$

$\theta_{\bar{r}-\bar{r}'} \equiv$ angle between $(x-x', y-y')$ and the x axis

t parametrizes the anisotropy between space and time directions

In the original Hamiltonian t is zero but it will be generated during renormalization due to the doping. The anisotropy parameter is first order in the interaction constants, and hence in the calculation of $R(r_1-r_2)$ we may reduce to the zeroth order, i.e. for $R(r_1-r_2)$ we may simply take $F(\bar{r}-\bar{r}') = \frac{1}{2}\ln\left(\frac{|\bar{r}-\bar{r}'|^2}{\alpha^2}\right)$.

- Let's calculate $R(r_1-r_2)$ (we take $a=1$). We will follow similar reasonings as in p. (56):

$$* \; R(r_1-r_2) = e^{-K F(r_1-r_2)} \left\{ 1 - \frac{2g^2}{(2\pi\alpha)^4 u^2} \int d^2R \int d^2r \, e^{-4KF(r)} \cos\delta x \; \times \right.$$
$$\left. \times \left\{ \begin{array}{l} x^2 \left[F(r_1-R)-F(r_2-R)\right] \partial_X^2 \left[F(r_1-R)-F(r_2-R)\right] \\ + y^2 \left[F(r_1-R)-F(r_2-R)\right] \partial_Y^2 \left[F(r_1-R)-F(r_2-R)\right] \end{array} \right\} \right\}$$

let $I_\pm(\bar r_1-\bar r_2) = -\frac{1}{2}\int d^2R \left[F(r_1-R)-F(r_2-R)\right]\left(\partial_X^2 \pm \partial_Y^2\right)\left[F(r_1-R)-F(r_2-R)\right]$

$$J_\pm(\delta) = \int \frac{d^2r}{\alpha^4}\,(x^2 \pm y^2)\cos\delta x \; e^{-4KF(r)}$$

Then:

$$R(r_1-r_2) = e^{-KF(r_1-r_2)} \left\{ 1 + \frac{(g/\pi u)^2}{8\pi^2} K^2 \left[J_+(\delta)\, I_+(\bar r_1-\bar r_2) + J_-(\delta)\, I_-(\bar r_1-\bar r_2)\right] \right\}$$

let's evaluate $J_\pm$ and $I_\pm$

- $I_+(\bar r_1-\bar r_2) = -\frac{1}{2}\int d^2R \left(F(r_1-R)-F(r_2-R)\right)\left(\partial_X^2+\partial_Y^2\right)\left[F(r_1-R)-F(r_2-R)\right]$

p. (57) $\searrow$ $= 2\pi F(r_1-r_2) = \boxed{2\pi \log\left(\frac{|\bar r_1-\bar r_2|}{\alpha}\right) = I_+(\bar r_1-\bar r_2)}$

- $I_-(\bar r_1-\bar r_2) \overset{\text{by parts}}{=} +\frac{1}{2}\int d^2R \left[\left(\partial_X F(r_1-R)\right)^2 - \left(\partial_Y F(r_1-R)\right)^2\right]$

$$+\frac{1}{2}\int d^2R \left[\left(\partial_X F(r_2-R)\right)^2 - \left(\partial_Y F(r_2-R)\right)^2\right]$$
$$- \int d^2R \left[\partial_X F(r_1-R)\,\partial_X F(r_2-R) - \partial_Y F(r_1-R)\,\partial_Y F(r_2-R)\right]$$

Since $\int d^2R \left(\left[\partial_X F(r_j-R)\right]^2 - \left(\partial_Y F(r_j-R)\right)^2\right) \underset{\vec R \leftarrow \bar r_j - \bar R}{\overset{\uparrow}{=}} \int d^2R \left(\frac{X^2-Y^2}{(X^2+Y^2)^2}\right) = 0$

and substituting:

$$\bar r_1 - \bar R = \left(\frac{\bar r_1-\bar r_2}{2}\right) + \left(\frac{\bar r_1+\bar r_2}{2} - \bar R\right) = \bar R' + (\bar r_1-\bar r_2)/2 \equiv \bar R' + \bar r$$

$$\bar r_2 - \bar R = \left(\frac{\bar r_2-\bar r_1}{2}\right) + \left(\frac{\bar r_1+\bar r_2}{2} - \bar R\right) = \bar R' - (\bar r_1-\bar r_2)/2 \equiv \bar R' - \bar r$$

we get that

$$I_- = -\int d^2R \left[\partial_X f(\bar R + \bar r)\, \partial_X f(\bar R - \bar r) - \partial_Y f(\bar R + \bar r)\, \partial_Y f(\bar R - \bar r) \right]$$

$$= -\int d^2R \left\{ \frac{(X+x)(X-x) - (Y+y)(Y-y)}{(\bar R + \bar r)^2 (\bar R - \bar r)^2} \right\}$$

$$= -\int d^2R \left\{ \frac{(X^2 - Y^2)}{(R^2-r^2)^2} - \frac{(x^2-y^2)}{(R^2-r^2)^2} \right\}$$

$$= (x^2-y^2) \int \frac{d^2R}{(R^2-r^2)^2} = \pi (x^2-y^2) \int_0^\infty \frac{2R\,dR}{(R^2-r^2)^2} \overset{s = \frac{R^2}{r^2} - 1}{\downarrow}{=} \frac{\pi (x^2-y^2)}{r^2} \int_{-1}^{\infty} \frac{ds}{s^2}$$

$$= \pi \left(\frac{x^2-y^2}{r^2} \right) = \pi \left[\frac{(x_1-x_2)^2 - (y_1-y_2)^2}{|\bar r_1 - \bar r_2|^2} \right]$$

Hence $\boxed{I_- = \pi \cos 2\theta_{\bar r_1 - \bar r_2}}$

- $$J_+(\delta) = \int \frac{d^2r}{\alpha^4} (x^2 + y^2) \cos \delta x \; e^{-4kf(r)}$$

$$= \int_\alpha^\infty \frac{dr\, r^3}{\alpha^4} \int_0^{2\pi} d\phi \, \cos(\delta r \cos\phi) \left(\frac{\alpha}{r} \right)^{4k}$$

$$= \boxed{2\pi \int_\alpha^\infty \frac{dr}{\alpha} \left(\frac{\alpha}{r} \right)^{4k-3} J_0(\delta r) = J_+(\delta)}$$

- $$J_-(\delta) = \int \frac{d^2r}{\alpha^4} (x^2 - y^2) \cos \delta x \; e^{-4kf(r)}$$

$$= \int_\alpha^\infty \frac{dr}{\alpha} \left(\frac{\alpha}{r} \right)^{4k-3} \int_0^{2\pi} d\phi \, \cos 2\phi \, \cos(\delta r \cos\phi)$$

$$= \boxed{-2\pi \int_\alpha^\infty \frac{dr}{\alpha} \left(\frac{\alpha}{r} \right)^{4k-3} J_2(\delta r) = J_-(\delta)}$$

* Re-exponentiating the expression of $R(r_1-r_2)$ in p. (92) we get: (94)

$$R(r_1-r_2) = e^{-\frac{K}{2}\log\left(\frac{(\bar r_1-\bar r_2)^2}{\alpha^2}\right)+\frac{K^2}{2}\left[\frac{(g/\pi u)^2}{8\pi^2}\cdot 4\pi^2\int_\alpha^\infty \frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4K} J_0(\delta r)\right]\log\left(\frac{(\bar r_1-\bar r_2)^2}{\alpha^2}\right)}$$

$$\cdot\, e^{-t\cos 2\theta_{\bar r_1-\bar r_2}-\left[\frac{(g/\pi u)^2}{8\pi^2}K^2\cdot 2\pi^2\int_\alpha^\infty \frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4K} J_2(\delta r)\right]\cos\theta_{\bar r_1-\bar r_2}}$$

we get hence an effective K_{eff} and an effective t_{eff}:

$$K_{eff} = K - K^2\frac{y^2}{2}\int_\alpha^\infty \frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4K} J_0(\delta r)$$

$$t_{eff} = t + K^2\frac{y^2}{4}\int_\alpha^\infty \frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4K} J_2(\delta r)$$

. let's shift the cut-off of the lowest limit of the integral here the procedure is the same as in p. (58):

$$K_{eff} = K - K^2\frac{y^2}{2}\frac{d\alpha}{\alpha}J_0(\delta\alpha) - K^2\frac{y^2}{2}\int_{\alpha+d\alpha}^\infty \frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4K} J_0(\delta r)$$

$$= \left[K - K^2\frac{y^2}{2}\frac{d\alpha}{\alpha}J_0(\delta\alpha)\right] - \frac{K^2}{2}\left[y^2\left(\frac{\alpha+d\alpha}{\alpha}\right)^{4-4K}\right]\int_{\alpha+d\alpha}^\infty \frac{dr}{\alpha+d\alpha}\left(\frac{r}{\alpha+d\alpha}\right)^{3-4K} J_0(\delta r)$$

Since K_{eff} gives the asymptotic properties it must be preserved by the cut-off hence we get the RG eqs:

$$K(\alpha+d\alpha) = K(\alpha) - K(\alpha)^2\frac{y(\alpha)^2}{2}J_0(\delta\alpha)\frac{d\alpha}{\alpha}$$

$$y(\alpha+d\alpha) = y(\alpha)\left(\frac{\alpha+d\alpha}{\alpha}\right)^{2-2K}$$

Note:
(recall that we always assume that we are perturbative in y)

let $\alpha = \alpha_0 e^{l}$, and $\delta(l) = \delta e^{l}$, then:

$$\frac{dK}{dl} = -K^2\, y^2/2\; J_0[\delta(l)\alpha_0]$$

$$\frac{dy}{dl} = (2-2K(l))y(l)$$

→ We consider first the case of constant doping, we'll see in a moment what happens for constant chemical potential

On the other hand:

$$t_{eff} = t + \frac{\pi^2 y^2}{4}\frac{d\alpha}{\alpha} J_2(\delta\alpha) + \frac{\pi^2}{2}\left[y^2\left(\frac{\alpha+d\alpha}{\alpha}\right)^{4-4K}\right]\int_{\alpha+d\alpha}^{\infty}\frac{dr}{\alpha+d\alpha}\left(\frac{r}{\alpha+d\alpha}\right)^{3-4K} J_2(\delta r)$$

Hence

$$t(\alpha+d\alpha) = t(\alpha) + \frac{\pi(\alpha)^2 y(\alpha)^2}{4} J_2(\delta\alpha)\frac{d\alpha}{\alpha}$$

and then:

$$\frac{dt}{dl} = \frac{1}{4}\pi^2 y^2 J_2[\delta(l)\alpha_0]$$

- Note that <u>regularizing M t is like renormalizing the velocity u.</u>

Let's see this

Say that we renormalize t $\Rightarrow F(r) = \frac{1}{2}\ln\left(\frac{x^2+u^2\tau^2}{\alpha^2}\right) + \frac{(t+\Delta t)}{K}\underbrace{\left[1-\frac{2u^2\tau^2}{x^2+u^2\tau^2}\right]}_{\cos\theta_r}$

$$= \frac{1}{2}\ln\left(\frac{x^2+u^2\tau^2}{\alpha^2}\right) - \frac{2\Delta t\, u^2\tau^2}{K(x^2+u^2\tau^2)} + \frac{t}{K}\left[1-\frac{2u^2\tau^2}{x^2+u^2\tau^2}\right] + \frac{\Delta t}{K}$$

Recall that

$$\frac{1}{2}\ln\left(\frac{x^2+(u+\Delta u)^2\tau^2}{\alpha^2}\right) \simeq \frac{1}{2}\ln\left(\frac{x^2+u^2\tau^2}{\alpha^2}\right) + \frac{u\Delta u\tau^2}{x^2+u^2\tau^2}$$

Hence to renormalize $t \to t+\Delta t$ is like renormalize $u \to u+\Delta u$ with $\Delta u = -\frac{2\Delta t\, u}{K}$

(<u>Note</u>: we are perturbative in t as well)

Hence: $\frac{dt}{dl} = -\frac{K}{2u}\frac{du}{dl}$

$\Delta u \to du$
$\Delta t \to dt$

* In this way we get the final RG eqs which are:

$$\left.\begin{cases} \dfrac{dK}{dl} = -\dfrac{1}{2} y^2 K^2 J_0[\delta(l)\alpha] \\ \dfrac{dy}{dl} = (2-2K)y \\ \dfrac{du}{dl} = -\dfrac{y^2}{2} u K J_2[\delta(l)\alpha] \end{cases}\right\} \equiv \underline{\text{RG eqs.}}$$

* So now there's a renormalization of the velocity because now time and space are not equivalent. Note however that the first two equations are independent of the last one, and hence the renormalization of u is actually not so important.

. Note also the presence of Bessel functions, and in particular of $J_0(\delta(l)\alpha)$ in the equation of dK/dl. This function obviously oscillates, but this is not important (it comes from the sharp cut-off employed, smoother cut-offs won't give oscillatory terms).

•What is important is that roughly $J_0 \sim 1$ for $\delta\alpha < 1$ and $J_0 \sim 0$ for $\delta\alpha > 1$. Hence, the renormalization of K due to the sine-Gordon cosine is suppressed for ~~[illegible]~~ lengthscales x such that $\delta x > 1$.

•Hence below $x \sim 1/\delta$ one may take $dK/dl \simeq -\frac{1}{2} y^2 K^2$, and one recovers the commensurate eqs.

·And beyond $x \sim 1/\delta$ one can forget the sine-gordon cosine, recovering a quadratic Hamiltonian with renormalized u^* and K^*.

* This behavior is physically quite easy to understand if we have a look to the term $\cos[\sqrt{8}\phi - \delta x]$. If $\delta x \ll 1$, we recover $\cos(\sqrt{8}\phi)$ and we may forget doping. But if $\delta x \gg 1$ then the cosine oscillates a lot averaging to zero, and we recover a Luttinger liquid.

·Hence: $\dfrac{1}{\delta} \cong$ distance at which the system "realizes" that it isn't commensurate!

- If we work at constant chemical potential (and not at constant doping) we should also look at its renormalization. Let's see how this is done.

Recall from p. (84) that adding the chemical potential to the Hamiltonian leads to:

$$H + \frac{\mu\sqrt{2}}{\pi}\int dx\,(\nabla\phi) = \frac{1}{2\pi}\int dx\left[uK(\nabla\theta)^2 + \frac{u}{K}(\nabla\phi)^2\right] + \frac{2g}{(2\pi\alpha)^2}\int dx\cos(\sqrt{8}\phi) + \frac{\mu\sqrt{2}}{\pi}\int dx\,\nabla\phi$$

$$= \frac{1}{2\pi}\int dx\left[uK(\nabla\theta)^2 + \frac{u}{K}(\nabla\tilde{\phi})^2\right] - \frac{\mu^2 KL}{\pi u} + \frac{2g}{(2\pi\alpha)^2}\int dx\cos\left[\sqrt{8}\tilde{\phi} - \frac{4K\mu}{u}x\right]$$

Where $\tilde{\phi} = \phi + \frac{K\mu}{u}\sqrt{2}\,x$

- We may evaluate the doping from the free energy using

$$\delta = -2\pi\frac{1}{L}\frac{\partial F}{\partial\mu} \qquad \text{with } F = \frac{-\log Z}{\beta} \implies \delta = \frac{2\pi}{L}\frac{1}{Z}\frac{\partial Z}{\partial\mu}$$

(Note: recall from p. (84) that $\delta = 2\pi d$, and d is the filling on top of half filling)

- The action is:

$$S = \underset{\substack{\uparrow\\ \text{quadratic part.}}}{S_0} - \frac{\mu^2 KL\beta}{\pi u} + \frac{2g}{(2\pi\alpha)^2}\int_0^\beta d\tau\int dx\cos\left[\sqrt{8}\tilde{\phi} - \frac{4K\mu}{u}x\right)$$

Hence:

$$Z = \int D\tilde{\phi}\; e^{-S_0}\, e^{\frac{\mu^2 KL\beta}{\pi u}}\, e^{-\frac{2g}{(2\pi\alpha)^2}\int_0^\beta d\tau\int dx\,\cos\left[\sqrt{8}\tilde{\phi} - \frac{4K\mu}{u}x\right)}$$

$$\frac{1}{Z}\frac{\partial Z}{\partial\mu} = \frac{2\mu KL\beta}{\pi u} - \frac{2gK}{\pi^2\alpha^2 u}\int_0^\beta d\tau\int dx\; x\left\langle \sin\left(\sqrt{8}\tilde{\phi} - \frac{4K\mu}{u}x\right)\right\rangle$$

Hence

$$\delta = \delta_Q - \frac{4gK}{u\pi\alpha^2}\frac{1}{\beta L}\int_0^\beta d\tau\int dx\; x\left\langle \sin\left[\sqrt{8}\tilde{\phi} - \delta_Q x\right]\right\rangle$$

where $\delta_Q = \frac{4K\mu}{u}$ is the value that δ would take for $g=0$, i.e. for a purely quadratic Hamiltonian.

• let's evaluate δ:

$$\langle \sin[\sqrt{8}\tilde{\phi}(x_1)-\delta_Q x_1]\rangle = \frac{1}{Z}\int D\tilde{\phi}\, \sin[\sqrt{8}\tilde{\phi}(x_1)-\delta_Q x_1]\, e^{-S_0}\, e^{-\frac{2g}{(2\pi\alpha)^2 u}\int d^2r_2 \cos[\sqrt{8}\tilde{\phi}(r_2)-\delta_Q x_2]}$$

up to 1st order

$$Z = \int D\tilde{\phi}\, e^{-S_0}\, e^{-\frac{2g}{(2\pi\alpha)^2 u}\int d^2r \cos[\sqrt{8}\tilde{\phi}(r)-\delta_Q x]} = \simeq Z_0\left[1 - \frac{2g}{(2\pi\alpha)^2 u}\int d^2r\, \cancelto{0}{\langle \cos[\sqrt{8}\tilde{\phi}(r)-\delta_Q x]\rangle_0} + \dots\right] = Z_0$$

$$\int D\tilde{\phi}\, \sin[\sqrt{8}\hat{\phi}(x_1)-\delta_Q x_1]\, e^{-S_0}\, e^{-\frac{2g}{(2\pi\alpha)^2 u}\int d^2r_2 \cos[\sqrt{8}\tilde{\phi}(r_2)-\delta_Q x_2)}$$

$$\simeq Z_0\left[\cancelto{0}{\langle \sin[\sqrt{8}\hat{\phi}(x_1)-\delta_Q x_1]\rangle_0} - \frac{2g}{(2\pi\alpha)^2 u}\int d^2r_2 \langle \sin(\sqrt{8}\hat{\phi}(x_1)-\delta_Q x_1)\cdot \cos(\sqrt{8}\hat{\phi}(r_2)-\delta_Q x_2)\rangle_0\right]$$

and:

$$\langle \sin[\sqrt{8}\hat{\phi}(r_1)-\delta_Q x_1)\cos(\sqrt{8}\hat{\phi}(r_2)-\delta_Q x_2]\rangle_0 =$$

$$= \frac{1}{4i}\langle\left[e^{i\sqrt{8}\hat{\phi}(r_1)}e^{-i\delta_Q x_1} - e^{-i\sqrt{8}\hat{\phi}(r_1)}e^{i\delta_Q x_1}\right]\left[e^{i\sqrt{8}\hat{\phi}(r_2)}e^{-i\delta_Q x_2} + e^{-i\sqrt{8}\hat{\phi}(r_2)}e^{+i\delta_Q x_2}\right]\rangle_0$$

$$= \frac{1}{4i}\left\{\langle e^{i\sqrt{8}[\hat{\phi}(r_1)-\hat{\phi}(r_2)]}\rangle_0\, e^{-i\delta_Q(x_1-x_2)} - \langle e^{-i\sqrt{8}(\hat{\phi}(r_1)-\hat{\phi}(r_2))}\rangle_0\, e^{i\delta_Q(x_1-x_2)}\right\} =$$

$$= \frac{-1}{2}\, e^{-4\langle(\hat{\phi}(r_1)-\hat{\phi}(r_2))^2\rangle_0}\sin\delta_Q(x_1-x_2) = \frac{-1}{2}\, e^{-4KF(\bar{r}_1-\bar{r}_2)}\sin\delta_Q(x_1-x_2)$$

• Hence

$$\langle \sin(\sqrt{8}\hat{\phi}(x_1)-\delta_Q x_1)\rangle = \frac{g}{(2\pi\alpha)^2 u}\int d^2r_2\, e^{-4F(\bar{r}_1-\bar{r}_2)}\sin\delta_Q(x_1-x_2)$$

and then:

$$\delta = \delta_Q - \frac{4gK}{u\pi\alpha^2}\,\frac{1}{\beta L}\,\frac{g}{(2\pi\alpha)^2 u^2}\int d^2r_1\int d^2r_2\, e^{-4KF(\bar{r}_1-\bar{r}_2)}\sin\delta_Q(x_1-x_2)\, x_1 \underset{\bar{R}=(\bar{r}_1+\bar{r}_2)/2,\ \bar{r}=\bar{r}_1-\bar{r}_2}{=} \delta_Q - \frac{Kg^2}{u^3\pi^3\alpha^4}\,\frac{1}{\beta L}\int d^2R\int d^2r\left(X+\frac{x}{2}\right)e^{-4KF(\bar{r})}\sin\delta_Q x \underset{\int d^2R=\beta L u}{=}$$

$$= \delta_Q - \frac{K g^2}{u^2 \pi^3 \alpha^4} \frac{1}{\beta L} \cdot \frac{L\beta}{2} \int d^2 r \; x \sin \delta_Q x \left(\frac{r}{\alpha}\right)^{-4K}$$

$$= \delta_Q - \frac{K g^2}{u^2 \pi^3 \alpha^4} \frac{1}{2} \int r^2 dr \left(\frac{r}{\alpha}\right)^{-4K} \underbrace{\int_0^{2\pi} d\beta \cos\beta \sin[\delta_Q r \cos\beta]}_{2\pi J_1(\delta_Q r)}$$

$$= \delta_Q - K \left(g/\pi u\right)^2 \int_\alpha^\infty \frac{dr}{\alpha} \left(\frac{r}{\alpha}\right)^{2-4K} \frac{J_1(\delta_Q r)}{\alpha}$$

Hence $\delta = \delta_Q - K y^2 \int_\alpha^\infty \frac{dr}{\alpha} \left(\frac{r}{\alpha}\right)^{2-4K} \frac{J_1(\delta_Q r)}{\alpha}$

- To change the cut-off is equivalent to re-scale $\delta(\ell) = e^\ell \delta$.

Hence $\delta(\ell + d\ell) = \delta(\ell) + \delta(\ell) d\ell$

$$\delta(\ell) = \delta_Q - K y^2 \frac{d\alpha}{\alpha} \frac{J_1(\delta_Q \alpha)}{\alpha} - K y^2 \int_{\alpha'}^\infty \frac{dr}{\alpha'} \left(\frac{r}{\alpha'}\right)^{2-4K} \frac{J_1(\delta_Q r)}{\alpha'} \left(\frac{\alpha'}{\alpha}\right)^{4-4K}$$

$$\delta' = \delta_Q' - K' y'^2 \int_{\alpha'}^\infty \frac{dr}{\alpha'} \left(\frac{r}{\alpha'}\right)^{2-4K} \frac{J_1(\delta_Q' r)}{\alpha'} \quad = \delta + \delta \, d\ell$$

Hence $\delta_Q - K y^2 \overset{d\ell}{\boxed{\frac{d\alpha}{\alpha}}} \frac{J_1(\delta_Q \alpha)}{\alpha} + \delta_Q d\ell = \delta_Q(\ell + d\ell)$

$$\longrightarrow \boxed{\frac{d\delta_Q}{d\ell} = \delta_Q(\ell) - \underbrace{K y^2 \frac{J_1(\delta_Q(\ell)\alpha)}{\alpha}}}$$

This comes from the chemical potential

→ It increase δ_Q

$\delta_Q(\ell) \sim \delta_Q e^\ell$

→ This comes from the cosine term

It reduces δ_Q

→ it wants to keep ϕ close to the minimum of the cosine

- On the other hand, when calculating the RG eqs. of p. (96) we will have $\delta_Q(\ell)$ instead of $\delta(\ell)$ in the arguments of the Bessel functions.

· For constant doping, $\delta(\ell) = e^{\ell}\delta_0$ was in the arguments of the Bessel functions. Hence, as mentioned before, the $J_0(\delta\alpha)$ term cuts the flow at some lengthscale ($> 1/\delta$). Hence as soon as $\delta_0 \neq 0$, the system is metallic, and the cosine term irrelevant.

· Now the situation is clearly different, since $\delta_\Omega(\ell)$ doesn't grow as e^{ℓ} due to the extra term in the equation in p. ⑲. the chemical potential grows δ_Ω whereas the cosine tends to decrease δ_Ω, and in turn δ_Ω tends to stop the RG flow of the cosine. Which wins in this competition depends on the value of the chemical potential. One hence expect two phases depending on μ.

· However, close to the transition ($\delta_\Omega \to 0$) the flow goes to strong coupling (recall that we always assume weak coupling in our calculations) before the renormalization of the cosine is stopped.

* A possible way of getting around this problem is provided by the Luther-Emery trick. Let's consider the Hamiltonian of p. ㊳

$$\tilde{K} = 2n^2 K_\rho = e^{-2\beta}$$

$$H = \frac{1}{2\pi}\int dx \left[u\tilde{K}(\nabla\tilde{\Theta})^2 + \frac{u}{\tilde{K}}(\nabla\tilde{\Phi})^2\right] + g_{1/2n}\int dx \cos(2\tilde{\Phi}) \underline{\underline{=}}$$

$$= \frac{1}{2\pi}\int dx\, u\left[e^{-2\beta}(\nabla\tilde{\Theta})^2 + e^{2\beta}(\nabla\tilde{\Phi})^2\right] + g_{1/2n}\int dx \cos(2\tilde{\Phi})$$

· We do now a trick

$$\frac{1}{2\pi}\int dx\, u\left[e^{-2\beta}(\nabla\tilde{\Theta})^2 + e^{2\beta}(\nabla\tilde{\Phi})^2\right] = \frac{v}{2\pi}\int dx\left[(\nabla\tilde{\Theta})^2 + (\nabla\tilde{\Phi})^2\right] + A$$

with $v = u(\operatorname{ch}2\beta + f_1 \operatorname{sh}2\beta)$ with f_1 in principle arbitrary and A something we have yet to determine. Note that v now acts as the velocity for an ideal gas.

- It's easy to see that A must be:

$$A = \frac{+u\,\mathrm{sh}2\beta}{2\pi}\int dx\,\left[-(1+f_1)(\nabla\tilde{\Theta})^2 + (1-f_1)(\nabla\tilde{\Phi})^2\right]$$

Recall (p.⑬) that $\nabla\tilde{\Phi} = -\pi(\rho_R+\rho_L)$, $\nabla\tilde{\Theta} = \pi(\rho_R-\rho_L)$

Then:

$$A = \frac{u\,\mathrm{sh}2\beta}{2\pi}\int dx\; 2\pi^2\left[2\rho_R\rho_L - f_1(\rho_R^2+\rho_L^2)\right] \overset{\text{Fourier-transforming}}{=}$$

$$= \frac{\pi u\,\mathrm{sh}2\beta}{L}\sum_p \left[2\rho_R(p)\rho_L(-p) - f_1\left[\rho_R(p)\rho_R(-p) + \rho_L(p)\rho_L(-p)\right]\right]$$

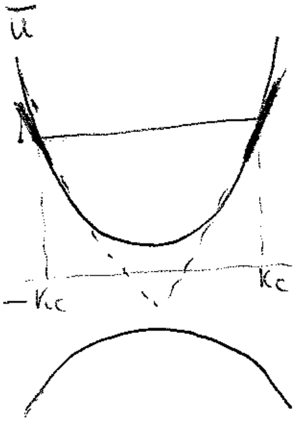
- Let $H_0 = \frac{v}{2\pi}\int dx\,((\nabla\tilde{\Theta})^2 + (\nabla\tilde{\Phi})^2) + g_{1/2D}\int dx\,\cos(2\tilde{\Phi})$

⇒ This is the Hamiltonian for the non-interacting case with back-scattering, i.e. like for the Luther-Emery point.

Hence:

$$H = H_0 + \frac{\pi u\,\mathrm{sh}2\beta}{L}\sum_p \left[2\rho_R(p)\rho_L(-p) - f_1\left[\rho_R(p)\rho_R(-p) + \rho_L(p)\rho_L(-p)\right]\right]$$

- To be able to dop, the chemical potential must be larger than the gap between the $\bar{u}$ and $\bar{e}$ bands of p.(89). Hence away from half-filling the chemical potential is in the upper band and in addition to the $\bar{e}$ states the $\bar{u}$ states are occupied up to a value $k_c = \pi d$ (where as before d is the doping).



Since we are only interested in the very low frequency behaviour we can linearize again around k_c.

$$\sqrt{(vk)^2+\Delta^2} \simeq \sqrt{(vk_c)^2+\Delta^2} + \begin{cases} v_c(k-k_c) & k \approx k_c \\ -v_c(k+k_c) & k \approx -k_c \end{cases}$$

with the new velocity $\boxed{v_c = \frac{v^2 k_c}{\sqrt{(vk_c)^2+\Delta^2}}}$

· We thus have to deal with a new Luttinger liquid of holons in the upper band. This will describe correlation functions at a distance larger than the inter-holon distance.

· Recall that

$$C^+_{\bar u k} = \alpha_k C^+_{Rk} + \beta_k C^+_{Lk} \qquad \text{with} \qquad \alpha_k = \left[\frac{1}{2} + \frac{vk}{2}\,\frac{1}{\sqrt{(vk)^2+\Delta^2}}\right]^{1/2}$$

$$C^+_{\ell k} = -\beta_k C^+_{Rk} + \alpha_k C^+_{Lk} \qquad\qquad \beta_k = \left(\frac{1}{2} - \frac{vk}{2}\,\frac{1}{\sqrt{(vk)^2+\Delta^2}}\right)^{1/2}$$

Hence:
$$C^+_{Rk} = \alpha_k C^+_{\bar u k} - \beta_k C^+_{\ell k}$$
$$C^+_{Lk} = \beta_k C^+_{\bar u k} + \alpha_k C^+_{\ell k}$$

In the following we will keep only the part containing $\bar u$ operators.

· Recall that

$$\rho_R(-p) = \sum_k C^+_{R,k+p} C_{Rk} \simeq \sum_k \alpha_{k+p}\alpha_k\, C^+_{\bar u,k+p} C_{\bar u k}$$

$$\rho_L(-p) = \sum_k C^+_{L,k+p} C_{L,k} \simeq \sum_k \beta_{k+p}\beta_k\, C^+_{\bar u,k+p} C_{\bar u k}$$

· We introduce now right and left movers for the holons; we call them, respectively a_k and b_k. Hence (keeping only terms close to k_c)

$$\rho_R(-p) \simeq \alpha^2_{k_c} \sum_k a^+_{k+p} a_k + \alpha^2_{-k_c} \sum_k b^+_{k+p} b_k$$

We define
$$\bar\rho_R(-p) = \sum_k a^+_{k+p} a_k$$
$$\bar\rho_L(-p) = \sum_k b^+_{k+p} b_k$$

Hence:
$$\rho_R(-p) \cong \alpha^2_{k_c}\,\bar\rho_R(-p) + \alpha^2_{-k_c}\,\bar\rho_L(-p)$$

Note that $\alpha_{-k_c} = \beta_{k_c}$, and
$$\alpha^2_{k_c} = \frac{1}{2}\left(1 + v_c/\bar v\right)$$
$$\beta^2_{k_c} = \frac{1}{2}\left(1 - v_c/\bar v\right)$$

$$\rho_R(-p) \simeq \alpha_{k_c}^2 \,\bar{\rho}_R(-p) + \beta_{k_c}^2 \,\bar{\rho}_L(-p)$$

$$\rho_L(-p) \simeq \beta_{k_c}^2 \,\bar{\rho}_L(-p) + \alpha_{k_c}^2 \,\bar{\rho}_L(-p)$$

- On the other hand

$$H_0 \simeq \sum_k k v_c \left(a_k^\dagger a_k - b_k^\dagger b_k\right)$$

- The interaction part becomes

$$\frac{\pi u \sinh 2\theta}{4L} \sum_p \Big[2\Big[\left(1-(v_c/v)^2\right)\left(\bar{\rho}_R(p)\bar{\rho}_R(-p)+\bar{\rho}_L(p)\bar{\rho}_L(-p)\right) + \left(1+\frac{v_c}{v}\right)^2 \bar{\rho}_R(p)\bar{\rho}_L(-p) + \left(1-\frac{v_c}{v}\right)^2 \bar{\rho}_L(p)\bar{\rho}_R(-p)\Big]$$
$$- f_1\Big[\left(1+\frac{v_c}{v}\right)^2 \bar{\rho}_R(p)\bar{\rho}_R(-p) + \left(1-\frac{v_c}{v}\right)^2 \bar{\rho}_L(p)\bar{\rho}_L(-p) + \left(1-\left(\frac{v_c}{v}\right)^2\right)\left(\bar{\rho}_R(p)\bar{\rho}_L(-p) + \bar{\rho}_L(p)\bar{\rho}_R(-p)\right)\Big)$$
$$- f_1\Big[\left(1-\frac{v_c}{v}\right)^2 \bar{\rho}_R(p)\bar{\rho}_R(-p) + \left(1+\frac{v_c}{v}\right)^2 \bar{\rho}_L(p)\bar{\rho}_L(-p) + \left(1-\left(\frac{v_c}{v}\right)^2\right)\left(\bar{\rho}_R(p)\bar{\rho}_L(-p) + \bar{\rho}_L(p)\bar{\rho}_R(-p)\right)\Big)\Big]$$

$$= \frac{\pi u \sinh 2\theta}{4L} \sum_p \left[\begin{aligned} &2\left[\left(1-\left(\frac{v_c}{v}\right)^2\right) - f_1\left(1+\left(\frac{v_c}{v}\right)^2\right)\right]\left(\bar{\rho}_R(p)\bar{\rho}_R(-p) + \bar{\rho}_L(p)\bar{\rho}_L(-p)\right) \\ &+ 4\left[\left(1+\left(\frac{v_c}{v}\right)^2\right) - f_1\left(1-\left(\frac{v_c}{v}\right)^2\right)\right]\bar{\rho}_R(p)\bar{\rho}_L(-p) \end{aligned}\right]$$

- Since f_1 is in principle arbitrary we will use it to cancel the $\rho_r(p)\rho_r(-p)$ terms:

$$f_1 = \frac{1-(v_c/v)^2}{1+(v_c/v)^2} = \frac{1}{1+2(vk_c)^2/\Delta^2}$$

and then ~~the~~ the interaction Hamiltonian becomes:

$$2\frac{\pi u \sinh 2\theta}{L}\,\frac{(v_c/v)^2}{1+(v_c/v)^2}\sum_p \bar{\rho}_R(p)\bar{\rho}_L(-p)$$

$$= \frac{2\pi u \sinh 2\theta}{L} f(k_c) \sum_p \bar{\rho}_R(p)\bar{\rho}_L(-p)$$

where $f(k_c) = \dfrac{(vk_c)^2}{2(vk_c)^2+\Delta^2} \propto k_c^2$ for small doping

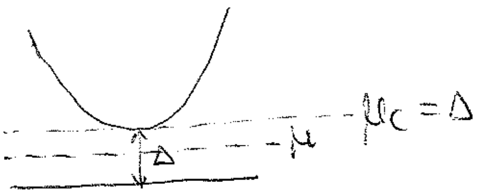
· Hence

$$H \cong \sum_k v_c k \left(a_k^\dagger a_k - b_k^\dagger b_k\right) + \frac{2\pi u \sinh 2\theta}{L} f(k_c) \sum_p \bar{\rho}_R(p)\,\bar{\rho}_L(-p)$$

· Note that since the residual interaction is proportional to k_c^2, sufficiently close to half filling the interaction term becomes negligible compared to the Fermi-Energy. This justifies the linearization of the spectrum, but it means as well that the holons behave as free particles when the doping is very small (Note: This result is valid regardless the strength of the original coupling)

In other words at small ~~the~~ doping the holons are very diluted and they don't see each other.

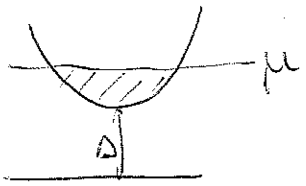
· We can ~~then~~ easily determine the doping as function of the chemical potential:



- For $\mu < \Delta \to$ no doping

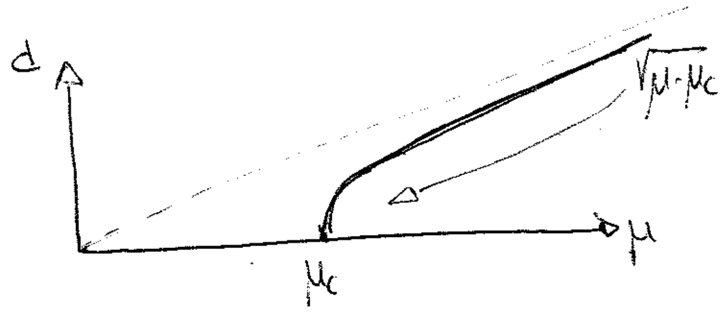
  Critical $\mu \to \mu_c = \Delta$

- For $\mu > \Delta \to$ doping given by the free-holon picture: $d = k_c/\pi$ with



$$\mu = \sqrt{(v k_c)^2 + \Delta^2}$$

$$\simeq \Delta\left[1 + \frac{(v k_c)^2}{2\Delta^2}\right] \longrightarrow \frac{v^2}{2\Delta^2} k_c^2 = \mu - \Delta \longrightarrow \boxed{d \propto \sqrt{\mu - \mu_c}}$$



* The fact that at very low doping the system behaves as ~~a~~ a non-interacting spin-less Fermi system (of holons) leads to <u>interesting consequences for the Mott-δ transition</u>:

* let's consider correlations of the form:

$$\tilde{\Phi} = n\sqrt{2}\phi$$

$$A(x) = \langle (\psi_R^\dagger \psi_L)_x (\psi_L^\dagger \psi_R)_x \rangle = \langle e^{i2\tilde{\Phi}(x)} e^{-i2\tilde{\Phi}(0)} \rangle \downarrow =$$

$$= \langle e^{+i2\sqrt{2}n(\phi(x)-\phi(0))} \rangle =$$

we consider that the original system may be described by an ~~original~~ Luttinger fixed point with K^* and u^*

$$\propto \left(\frac{\alpha}{x}\right)^{4n^2K^*}$$

On the other hand at $k_c \to 0$ the holons behave as non-interacting spinless fermions and hence (p. ㉙) one expects

$$A(x) \sim 1/x^2$$

Hence at the <u>Mott-δ transition</u> $(k_c \to 0)$: $K^* = \dfrac{1}{2n^2}$

Recall that at the <u>Mott-U transition</u> (p. (86)): $K^* = 1/n^2$

* Recall from p. (101) that the holon velocity is

$$v_c = \frac{v^2 k_c}{\sqrt{(vk_c)^2+\Delta^2}} \underset{k_c \to 0}{\sim} k_c/\Delta$$

Recall from p. ⑲ that the compressibility $K_2 \sim 1/v_c$.

Hence at the Mott-δ transition the <u>compressibility diverges</u> before dropping to zero in the Mott phase (p. (87)).

* For ~~a~~ free spinless fermions (at the bottom of the energy band) $\omega \sim k^2$. Recall (p. (87)) that the dynamical exponent (z) tells us how the characteristic "τ-length" relates to the characteristic x-length ($\xi_\tau \sim \xi_x^z \Rightarrow 1/\omega \sim 1/k^z$) $\Longrightarrow$ Hence for the Mott-δ transition $\boxed{z=2}$. Recall that for the Mott-U transition the dynamical exponent was $z=1$.