* EFFECTS OF MAGNETIC FIELD AND MAGNETIC ANISOTROPY

· We will now see what happens when the spin rotation invariance is broken. There're 2 main ways to do that

- Add a magnetic field
- Have spin-anisotropic interactions

We will see both possibilities in detail.

* Magnetic field

Let's consider a magnetic field B along the z-direction. The effects of the magnetic field enter in the Hamiltonian in the form:

$$H \sim -\frac{h}{2}\int dx\,(\rho_\uparrow - \rho_\downarrow) \underset{\substack{\uparrow \\ p.\,34\text{-}35}}{=} \frac{h}{\sqrt{2}}\int dx \frac{1}{\pi}\nabla\phi_\sigma(x)$$

where $h = g\mu_B B$. It looks very similar as the chemical pot. term but now with ϕ_σ instead of ϕ_ρ.

The total Hamiltonian for the spin sector is hence that of p. 38 + magnetic field:

$$H = H_\sigma^0 + \frac{2g_{1\perp}}{(2\pi\alpha)^2}\int dx \cos\sqrt{8}\phi_\sigma + \frac{h}{\sqrt{2}\pi}\int dx\, \nabla\phi_\sigma(x)$$

· As for the chemical potential we may re-absorb the magnetic-field term into H_σ^0: $\tilde{\phi}_\sigma = \phi_\sigma + \frac{hK_\sigma}{\sqrt{2}u_\sigma}x$

$$H = \frac{1}{2\pi}\int dx\,\left[u_\sigma K_\sigma(\nabla\theta_\sigma)^2 + \frac{u_\sigma}{K_\sigma}(\nabla\tilde{\phi}_\sigma)^2\right] + \frac{2g_{1\perp}}{(2\pi\alpha)^2}\int dx\cos\left[\sqrt{8}\,\tilde{\phi}_\sigma - \frac{2hK_\sigma}{u_\sigma}x\right]$$

· We face hence a similar problem as the one we faced in p. 90. We may perform the same RG treatment as we did there and get to the RG equations of p. 96. The role of δ is played here by $2hK_\sigma/u_\sigma$.

· Let's have a look first to the regime where the cosine term is marginal, and hence the RG eqs. remain valid at any length scale. I recall you that in the RG eqs, h doesn't couple with K and y. However there's a regularization introduced by the $J_0(\delta\alpha)$ term, which cuts the renormalization flow for $\alpha \sim 1/\delta$.

* Let's recall that at the separatrix ($y_{\parallel} = y$):

$$y = \frac{y_0}{1+y_0 l}$$

We start at the separatrix.

. The flow stops at the scale $\delta\alpha(l) \sim 1$. Since $\alpha(l) = \alpha_0 e^l$, then the flow stops for

$$l^* = \log\left(\frac{1}{\alpha_0 \delta}\right)$$

Thus, at the scale where the RG flow is cut, the interaction parameter becomes, for a weak enough magnetic field (δ small)

$$y(l^*) = \frac{y_0}{1+y_0 l^*} \simeq 1/l^* = 1/\log(1/\alpha_0\delta)$$

For weak B $\longrightarrow$ l^* is very large and hence $y(l^*)$ becomes independent of the original coupling. Note that $y(l^*)$ depends quite singularly on the magnetic field (recall that $\delta \sim B$).

. Let's evaluate the spin susceptibility (p. 39): $\chi_\sigma = \dfrac{K_\sigma}{2\pi u_\sigma}$.

Recall from p. 37 that for spin-isotropic interactions

$$\left.\begin{aligned} u_\sigma &= u\left(1-(y/2)^2\right)^{1/2} \\ K_\sigma &= \left[\frac{1+y/2}{1-y/2}\right]^{1/2} \end{aligned}\right\}$$

Note: In p. 39 we obtained that for the $g_{1\parallel} = 0$ case. Since at the scale l^* the cosine term is irrelevant, we may use this expression here

* From the RG eqs. of p. 46 we see that the renormalization of u is negligible for very small h ($du/dl \sim y^2 u$). Hence we just need to consider the renormalization of y. Hence:

$$\frac{K_\sigma(l^*)}{u_\sigma(l^*)} = \frac{1}{u}\,\frac{\left(\frac{1+y(l^*)/2}{1-y(l^*)/2}\right)^{1/2}}{\left(1-(y(l^*)/2)^2\right)^{1/2}} \simeq \frac{1}{u}\left(1+y(l^*)/2\right)$$

Hence:

$$\chi_\sigma = \frac{1}{2\pi u}\left[1+y(l^*)/2\right]$$

and consequently:

$$\boxed{k_\sigma = k_\sigma^0 + \frac{1}{4\pi u} \frac{1}{\log\left(\frac{u}{2uk\alpha_0}\right)}}$$

This is a very singular increase of k_σ with h. Note that $\frac{dk_\sigma}{dh}$ is infinite for $h \to 0$.

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* Magnetic anisotropies

· Magnetic anisotropy can come from spin-anisotropic interactions. We have already seen (p. 34) that we may have different coupling constants for interaction between particles of the same spin ($g_\parallel$) or of opposite spin ($g_\perp$).

· In the following we will analyze the most general coupling constants that describe spin-anisotropic interactions. In order to do so we will employ the operators for charge and spin density waves (p. 39 and 40)

$$O_i = \sum_{\sigma,\sigma'} \psi^+_{k_F,\sigma}(x)\, \sigma^{(i)}_{\sigma,\sigma'}\, \psi_{-k_F,\sigma'}(x) \qquad (\sigma^0 \equiv \mathbb{1}) \qquad i = 0, 1, 2, 3$$

We can use the tensors $O_i^+ O_j$ as a basis to represent the most general interaction term H_{int}

$$H_{int} = \sum_{\alpha,\beta} \int dx\, g_{\alpha\beta}\, O^+_\alpha(x)\, O_\beta(x)$$

If we choose the axis system where the tensor is diagonal (and without entering into too much detail) we get the most general form:

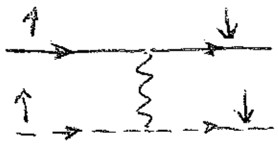
$$H_{int} = -\int dx \left\{ (g_{2\parallel} - g_{1\parallel} - g_{1\perp})\, O_0^+ O_0 + (g_{2\perp} + g_f)\, O_1^+ O_1 + (g_{2\perp} - g_f)\, O_2^+ O_2 + (g_{2\parallel} - g_{1\parallel} + g_{1\perp})\, O_3^+ O_3 \right\}$$

(the Hamiltonian must be time-reversal and inversion symmetric)

* Since there're only four terms, only four independent coefficients are needed to describe the interactions.

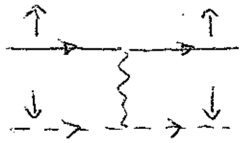
In the previous expressions we identify already known terms, namely $g_{1\parallel}, g_{1\perp}, g_{2\parallel}$ and $g_{2\perp}$. Recall that $g_{2\parallel}$ and $g_{1\parallel}$ describe actually the same process ($\tilde{g}_{2\parallel} = g_{2\parallel} - g_{1\parallel}$, p. 37).

Note that we need a fourth coupling constant (g_f) in addition to $g_{1\perp}, g_{2\perp}$ and $\tilde{g}_{2\parallel}$. The g_f processes are of the form



It looks like a g_2 process but now both spins are flipped

(Note: in the previous graph $\rightarrow \equiv k_F$, $- \rightarrow - \equiv -k_F$

Hence  $\Rightarrow g_2$  $\Rightarrow g_4$)

* From the previous equation we see clearly that
  * g_f represents the anisotropy in the xy plane ($g_{2\perp} + g_f$ vs. $g_{2\perp} - g_f$)
  * $\tilde{g}_{2\parallel} - g_{2\perp} + g_{1\perp}$ represents the uniaxial anisotropy along the z-direction

Note: if we substract $g_{2\perp}$ from all $O_1^\dagger O_1$, $O_2^\dagger O_2$ and $O_3^\dagger O_3$ couplings, we get, $+g_f$, $-g_f$, $\tilde{g}_{2\parallel} + g_{1\perp} - g_{2\perp}$, which are the constants mentioned above.)

* Since we've already studied the $g_{2\parallel}, g_{1\parallel}$ and $g_{2\perp}$ processes, let's have now a look to the g_f processes; these processes are of the form:

$$- g_f \int dx \underbrace{(\psi_{R\uparrow}^\dagger \psi_{L\downarrow})(\psi_{L\uparrow}^\dagger \psi_{R\downarrow})}_{\downarrow \text{ p. } 35 \quad \frac{e^{-i2\sqrt{2}\theta_\sigma}}{(2\pi\alpha)^2}} + \text{h.c.} = \frac{-2g_f}{(2\pi\alpha)^2} \int dx \cos(\sqrt{8}\,\theta_\sigma)$$

* Note that in the form of the g_f processes we wrote $(\psi_{R\uparrow}^\dagger \psi_{L\downarrow})(\psi_{L\uparrow}^\dagger \psi_{R\downarrow})$ and then bosonize, instead of a g_2-like form: $\psi_{R\uparrow}^\dagger \psi_{L\uparrow}^\dagger \psi_{L\downarrow} \psi_{R\downarrow}$. This is

a quite important point that we will discuss later. For the moment we take the expression above (which turns out to be the correct one) and then we will show why (it has to do with the Klein factors!).

· One thus have the full Hamiltonian (that of p. (38) plus g_f terms):

$$H = H_\rho^\circ + H_\sigma^\circ + \frac{2g_{1\perp}}{(2\pi\alpha)^2}\int dx \cos\sqrt{8}\,\phi_\sigma - \frac{2g_f}{(2\pi\alpha)^2}\int dx \cos\sqrt{8}\,\theta_\sigma$$

· We will derive now the RG equation following the same procedure as in p. (53):

$$\langle e^{i\sqrt{2}(\phi_1-\phi_2)}\rangle_H = \frac{1}{Z}\int D\phi\, D\theta\, e^{i\sqrt{2}(\phi_1-\phi_2)} e^{-S_0} e^{-\frac{2g_{1\perp}}{(2\pi\alpha)^2u}\int d^2r \cos\sqrt{8}\phi - \frac{2g_f}{(2\pi\alpha)^2u}\int d^2r\cos\sqrt{8}\theta}$$

As in p. (53):

$$Z \cong Z_0\left\{1 + \frac{1}{2}\left(\frac{2g_{1\perp}}{(2\pi\alpha)^2u}\right)^2\int d^2r\int d^2r' \langle\cos\sqrt{8}\phi\,\cos\sqrt{8}\phi'\rangle + \frac{1}{2}\left(\frac{2g_f}{(2\pi\alpha)^2u}\right)^2\int d^2r\int d^2r'\langle\cos\sqrt{8}\theta\,\cos\sqrt{8}\theta'\rangle\right\}$$

and

$$\langle e^{i\sqrt{2}(\phi_1-\phi_2)}\rangle_H = \frac{Z_0}{Z}\left[\langle e^{i\sqrt{2}(\phi_1-\phi_2)}\rangle_0 + \frac{1}{2}\left(\frac{2g_{1\perp}}{(2\pi\alpha)^2u}\right)^2\int d^2r\int d^2r'\langle e^{i\sqrt{2}(\phi_1-\phi_2)}\cos\sqrt{8}\phi\,\cos\sqrt{8}\phi'\rangle_0 \right.$$
$$\left. - \frac{1}{2}\left(\frac{2g_f}{(2\pi\alpha)^2u}\right)^2\int d^2r\int d^2r'\langle e^{i\sqrt{2}(\phi_1-\phi_2)}\cos\sqrt{8}\theta\,\cos\sqrt{8}\theta'\rangle_0\right]$$

then:

$$\langle e^{i\sqrt{2}(\phi_1-\phi_2)}\rangle_H \cong \langle e^{i\sqrt{2}(\phi_1-\phi_2)}\rangle_0 + \frac{1}{2}\left(\frac{2g_{1\perp}}{(2\pi\alpha)^2u}\right)^2\int d^2r\int d^2r'\left[\langle e^{i\sqrt{2}(\phi_1-\phi_2)}\cos\sqrt{8}\phi\,\cos\sqrt{8}\phi'\rangle_0 - \langle e^{i\sqrt{2}(\phi_1-\phi_2)}\rangle_0\langle\cos\sqrt{8}\phi\,\cos\sqrt{8}\phi'\rangle_0\right]$$
$$+ \frac{1}{2}\left(\frac{2g_f}{(2\pi\alpha)^2u}\right)^2\int d^2r\int d^2r'\left[\langle e^{i\sqrt{2}(\phi_1-\phi_2)}\cos\sqrt{8}\theta\,\cos\sqrt{8}\theta'\rangle_0 - \langle e^{i\sqrt{2}(\phi_1-\phi_2)}\rangle_0\langle\cos\sqrt{8}\theta\,\cos\sqrt{8}\theta'\rangle_0\right]$$

Using the expressions for the correlation functions $\langle\ldots\rangle_0$ (p. (27))

$$\langle e^{i\sqrt{2}(\phi_1-\phi_2)}\rangle_H = e^{-K_\sigma F_1(r_1-r_2)}\left\{1 + \frac{g_{1\perp}^2}{2(2\pi\alpha)^2u^2}\int d^2r'\int d^2r''\sum_{\epsilon=\pm1} e^{-4K_\sigma F(r'-r'')} \times \left[e^{2\epsilon K_\sigma[F_1(r_1-r')-F_1(r_1-r'')-F_1(r_2-r')+F_1(r_2-r'')]}-1\right]\right.$$
$$\left. + \frac{g_f^2}{2(2\pi\alpha)^2u^2}\int d^2r'\int d^2r''\sum_{\epsilon=\pm1} e^{-4K_\sigma^{-1}F(r'-r'')} \times \left[e^{2\epsilon[F_2(r_1-r')-F_2(r_1-r'')-F_2(r_2-r')+F_2(r_2-r'')]}-1\right]\right\}$$

using as always $R = (r' + r'')/2$, $r = r' - r''$:

$$\langle e^{i\sqrt{2}(\phi_1 - \phi_2)}\rangle_H = e^{-K_\sigma f_1(r_1 - r_2)} \times \Big\{ 1 +$$

$$+ \frac{g_{1\perp}^2}{2(2\pi\alpha)^4 u^2} \int d^2R \int d^2r \sum_{\epsilon = \pm 1} e^{-4K_\sigma f_1(r)} \Big[e^{2\epsilon K_\sigma (f_1(r_1 - R - r/2) - f_1(r_1 - R + \frac{r}{2}) - f_1(r_2 - R - \frac{r}{2}) + f_1(r_2 - R + \frac{r}{2}))} - 1 \Big]$$

$$+ \frac{g_f^2}{2(2\pi\alpha)^4 u^2} \int d^2R \int d^2r \sum_{\epsilon = \pm 1} e^{-4K_\sigma^{-1} f_1(r)} \Big[e^{2\epsilon (f_2(r_1 - R - r/2) - f_2(r_1 - R + \frac{r}{2}) - f_2(r_2 - R - \frac{r}{2}) + f_2(r_2 - R + \frac{r}{2}))} - 1 \Big] \Big\}$$

we perform the same gradient expansion as in p. 55

$$= e^{-K f_1(r_1 - r_2)} \times \Big\{ 1 + \frac{g_{1\perp}^2 K_\sigma^2}{(2\pi\alpha)^4 u^2} \int d^2R \int d^2r\, r^2 e^{-4K_\sigma f_1(r)} \Big[(\nabla_x (f_1(r_1 - R) - f_1(r_2 - R)))^2 + (\nabla_y (f_1(r_1 - R) - f_1(r_2 - R)))^2 \Big]$$

$$+ \frac{g_f^2}{(2\pi\alpha)^4 u^2} \int d^2R \int d^2r\, r^2 e^{-4K_\sigma^{-1} f_1(r)} \Big[(\nabla_x (f_2(r_1 - R) - f_2(r_2 - R)))^2 + (\nabla_y (f_2(r_1 - R) - f_2(r_2 - R)))^2 \Big] \Big\}$$

from the definitions of f_1 and f_2 (p. 23) one can easily see that f_1 and f_2 are the real and imaginary part of $\ln(y_\alpha - ix)$ (see e.g. Arfken Sec. 6.2). Hence we may use the Cauchy-~~Riemann~~ conditions which tell us that $\nabla_x f_1 = i \nabla_y f_2$ and $\nabla_y f_1 = -i \nabla_x f_2$

$$= e^{-K_\sigma f_1(r_1 - r_2)} \times \Big\{ 1 + \frac{g_{1\perp}^2 K_\sigma^2}{(2\pi\alpha)^4 u^2} \int d^2R \int d^2r\, r^2 e^{-4K_\sigma f_1(r)} \Big[(\nabla_x (f_1(r_1 - R) - f_1(r_2 - R)))^2 + (\nabla_y (f_1(r_1 - R) - f_1(r_2 - R)))^2 \Big]$$

$$- \frac{g_f^2}{(2\pi\alpha)^4 u^2} \int d^2R \int d^2r\, r^2 e^{-4K_\sigma^{-1} f_1(r)} \Big[(\nabla_x (f_1(r_1 - R) - f_1(r_2 - R)))^2 + (\nabla_y (f_1(r_1 - R) - f_1(r_2 - R)))^2 \Big] \Big\}$$

from now on we proceed exactly in the same way as in p. 56 and 57

$$\simeq e^{-F(r_1 - r_2)\Big[K_\sigma - \frac{g_{1\perp}^2 K_\sigma^2}{4\pi^3 u^2 \alpha^4} \int_{r > \alpha} d^2r\, r^2 e^{-4K_\sigma f_1(r)} + \frac{g_f^2}{4\pi^3 u^2 \alpha^4} \int_{r > \alpha} d^2r\, r^2 e^{-4K_\sigma^{-1} f_1(r)} \Big]}$$

Hence:

$$K_{eff} = K_\sigma - \frac{g_{1\perp}^2 K_\sigma^2}{2(\pi u)^2}\int_\alpha^{\infty} d\left(\frac{r}{\alpha}\right)\left(\frac{r}{\alpha}\right)^{3-4K_\sigma} + \frac{g_f^2}{2(\pi u)^2}\int_\alpha^\infty d\left(\frac{r}{\alpha}\right)\left(\frac{r}{\alpha}\right)^{3-4K_\sigma^{-1}}$$

Shifting the cut-off:

$$K_{eff} = K_\sigma(\alpha) - \left[\frac{g_{1\perp}^2 K_\sigma^2}{2(\pi u)^2} - \frac{g_f^2}{2(\pi u)^2}\right]\frac{d\alpha}{\alpha} - \frac{g_{1\perp}^2 K_\sigma^2}{2(\pi u)^2}\left(\frac{\alpha+d\alpha}{\alpha}\right)^{4-4K_\sigma}\int_{\alpha+d\alpha}^\infty d\left(\frac{r}{\alpha+d\alpha}\right)\left(\frac{r}{\alpha+d\alpha}\right)^{3-4K_\sigma}$$

$$+ \frac{g_f^2}{2(\pi u)^2}\left(\frac{\alpha+d\alpha}{\alpha}\right)^{4-4K_\sigma^{-1}}\int_{\alpha+d\alpha}^\infty d\left(\frac{r}{\alpha+d\alpha}\right)\left(\frac{r}{\alpha+d\alpha}\right)^{3-4K_\sigma^{-1}}$$

- This leads to the RG equations (we follow the same procedure as in p. 58)

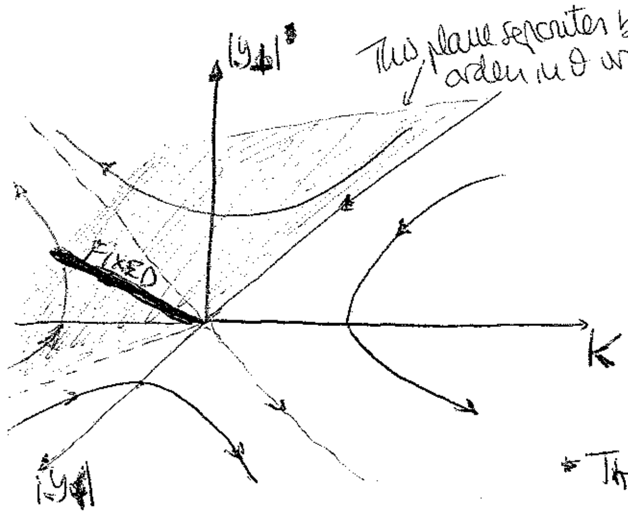
Defining as always $y_\perp = g_{1\perp}/\pi u$
$y_f = g_f/\pi u$:

$$\boxed{\begin{aligned} \frac{dK_\sigma}{dl} &= -\frac{1}{2}y_\perp^2 K_\sigma^2 + \frac{1}{2}y_f^2 \\ \frac{dy_\perp}{dl} &= (2-2K)y_\perp \\ \frac{dy_f}{dl} &= (2-2K^{-1})y_f \end{aligned}}$$

Note that these RG eqs are the same as those of p. 58 if we remove y_f.

* At the lowest order in interaction $K_\sigma \cong 1 + y_\sigma/2$ and then we get the RG eqs:

$$\boxed{\begin{aligned} dy_\sigma/dl &= y_f^2 - y_\perp^2 \\ dy_\perp/dl &= -y_\perp y_\sigma \\ dy_f/dl &= y_f y_\sigma \end{aligned}}$$



* Clearly the $\cos\sqrt{8}\theta$ term wants to order θ (and hence disorder φ) and $\cos\sqrt{8}\phi$ wants the opposite. Depending on the initial values a gap can open in the excitation of φ or θ (i.e. we get a flow towards $|y_\theta| \to \infty$ or $|y_\phi| \to \infty$)

* There's a critical line. For $K=1$, $y_f = y_\phi$ there's a fixed line of the flow for any point of the ~~plane~~ indicated plane → here the system remains critical (i.e. not massive)

. This fixed line may be understood as follows. Let's consider the Hamiltonian at the end of p. 103

$$H_{int} = -\int dx \left\{ (g_{2\parallel} - g_{1\parallel} - g_{1\perp}) O_0^\dagger O_0 + (g_{2\perp} + g_f) O_1^\dagger O_1 + (g_{2\perp} - g_f) O_2^\dagger O_2 + (g_{2\parallel} - g_{1\parallel} + g_{1\perp}) O_3^\dagger O_3 \right\}$$

Let's forget the charge part which is irrelevant for this discussion.

Extracting a constant we get

$$H_{int} = -\int dx \left[(g_\sigma + g_f) O_1^\dagger O_1 + (g_\sigma - g_f) O_2^\dagger O_2 + g_{1\perp} O_3^\dagger O_3 \right]$$

with $g_\sigma = g_{1\parallel} - g_{2\parallel} + g_{2\perp}$. Let's rotate around x to swap $y \leftrightarrow z$

$$H_{int} = -\int dx \left[(g_\sigma + g_f) O_1^\dagger O_1 + (g_{1\perp}) \tilde{O}_2^\dagger \tilde{O}_2 + (g_\sigma - g_f) \tilde{O}_3 \tilde{O}_3 \right]$$

$$= -\int dx \left[[C + G_\sigma + G_f] O_1^\dagger O_1 + [C + G_\sigma - G_f] \tilde{O}_2^\dagger \tilde{O}_2 + (C + G_{1\perp}] \tilde{O}_3^\dagger \tilde{O}_3 \right]$$

Let $C = -\frac{1}{2}(g_f + g_{1\perp} - g_\sigma)$ ← this constant is irrelevant and we'll remove it

Then:
$$\left. \begin{aligned} G_\sigma &= g_f + g_{1\perp} \\ G_f &= \tfrac{1}{2}(g_\sigma + g_f - g_{1\perp}) \\ G_{1\perp} &= \tfrac{1}{2}(g_\sigma - g_f + g_{1\perp}) \end{aligned} \right\} H_{int} = -\int dx \left[(G_\sigma + G_f) O_1^\dagger O_1 + (G_\sigma - G_f) \tilde{O}_2^\dagger \tilde{O}_2 + G_{1\perp} \tilde{O}_3^\dagger \tilde{O}_3 \right]$$

Note that for $g_{1\perp} = g_f \longrightarrow G_\sigma = 2g_f$, $G_f = G_{1\perp} = \frac{g_\sigma}{2}$; We can then remove the global constant $g_\sigma/2$ to get a system with $G_f = G_{1\perp} = 0 \Longrightarrow$ This would be a purely quadratic Hamiltonian (the cosines disappear!) with a given K_σ.

. Hence, the same operator which is rotation invariant in the fermionic language can give either quadratic terms or cosine terms. This reflects the fact that the bosonization formulas do not obviously exhibit the symmetries of the original model.
. This occurs because in our bosonization method we have chosen explicitely a quantization axis, so the method breaks the rotational symmetry in implementing the operators. The bosonization method assumed is hence valid for systems that don't have a large symmetry group. For many situations that's enough. There's however an alternative method known as <u>non-abelian bosonization</u> which respects symmetries (but it's much more cumbersome to use).

KLEIN FACTORS

* Let's come back to the <u>definition of the g_f term</u>. In p. 109 we defined these terms as:

$$-g_f \int dx \, (\psi^+_{R\uparrow} \psi_{L\downarrow})(\psi^+_{L\uparrow} \psi_{R\downarrow}) + h.c.$$

which after bosonization gave us $-2\frac{g_f}{(2\pi\alpha)^2}\int dx \cos(\sqrt{8}\,\theta_\sigma)$

* However we could have written the g_f term in the more familiar <u>g_2-like form</u> permuting two fermion operators:

$$g_f \int dx \, \psi^+_{R\uparrow} \psi^+_{L\uparrow} \psi_{L\downarrow} \psi_{R\downarrow} + h.c.$$

which after bosonization would give us $2\frac{g_f}{(2\pi\alpha)^2}\int dx \cos(\sqrt{8}\,\theta_\sigma)$, i.e. exactly the same as before but with a minus sign!
. Let's assume a regime where g_f is relevant, and hence θ_σ orders. If we use the new (wrong) expression, $g_f > 0$ means that the system orders for $\theta_\sigma = \pi/\sqrt{8}$ (whereas for the old correct expression the system order for $\theta_\sigma = 0$) (recall our discussion of p. 67-69).

· let's recall the bosonized form of O^x_{SDW} and O^y_{SDW} (p. 43 - 44):

$$O^x_{SDW}(x) = \frac{e^{-2ik_F x}}{\pi \alpha} e^{i\sqrt{2}\phi_\rho} \cos(\sqrt{2}\theta_\sigma)$$

$$O^y_{SDW}(x) = \frac{-e^{-2ik_F x}}{\pi \alpha} e^{i\sqrt{2}\phi_\rho} \sin(\sqrt{2}\theta_\sigma)$$

Recall that $O_{CDW}(x) \sim \cos\sqrt{2}\phi_\sigma$ and $O^z_{SDW}(x) \sim \sin\sqrt{2}\phi_\sigma$. Since θ_σ orders then both CDW and SDW_z correlation functions should now decay exponentially to zero (p. 67).

For the g_2-like form $\rightarrow \theta_\sigma \rightarrow \pi/\sqrt{8}$ $\rightarrow \cos(\sqrt{2}\theta_\sigma) \Rightarrow \cos\frac{\pi}{2} = 0$
$\rightarrow \sin(\sqrt{2}\theta_\sigma) \Rightarrow \sin\pi/2 = 1$

Hence O^x_{SDW} - correlations will exponentially decay to zero, whereas $O^y_{SDW}(x)$ - correlations will decay exponentially to a constant. The g_2-like form would hence predict a SDW along y.

· For the old (correct) way $\rightarrow \theta_\sigma \Rightarrow 0$ $\rightarrow \cos(\sqrt{2}\theta_\sigma) \rightarrow 1$
$\sin(\sqrt{2}\theta_\sigma) \rightarrow 0$

and hence we recover a SDW along x.

· It's easy to see that this latter is the correct one.

Recall from p. 108 that

$$H_{int} \sim -\int dx \, (g_f \underbrace{O_1^\dagger O_1}_{SDW_x} - g_f \underbrace{O_2^\dagger O_2}_{SDW_y})$$

Hence $g_f > 0$ tends to order SDW_x and not SDW_y (!!)

· Hence the g_2-like form is wrong.

It's wrong because in our bosonization we forgot the Klein factors (p. 11)

* Let's use now the Klein factors when bosonizing. Using the expression of p. 35 for the $g_{1\perp}$ and the g_f terms ($\psi^+_{R\uparrow}\psi^+_{L\downarrow}\psi_{R\downarrow}\psi_{L\uparrow}$ and $\psi^+_{R\uparrow}\psi^+_{L\uparrow}\psi_{R\downarrow}\psi_{R\downarrow}$, respectively) we get

$$H = H_0 + \frac{g_{1\perp}}{(2\pi\alpha)^2}\int dx\left[U^+_{R\uparrow}U^+_{L\downarrow}U_{R\downarrow}U_{L\uparrow}\, e^{i\sqrt{8}\phi_\sigma(x)} + h.c.\right]$$

$$- \frac{g_f}{(2\pi\alpha)^2}\int dx\left[U^+_{R\uparrow}U^+_{L\uparrow}U_{R\downarrow}U_{L\uparrow}\, e^{-i\sqrt{8}\theta_\sigma(x)} + h.c.\right]$$

(Note: In the thermodynamic limit we can forget that U and U^+ lower/raise the total number of particles, and only focus on the sign due to these operators). [Note II: Recall that U_r of different species anticommute, whereas for the same species $UU^+ = U^+U = 1$ (see p. 11)]

This form of the Hamiltonian must be compared with that of p. 110.

$$H = H_0 + \frac{g_{1\perp}}{(2\pi\alpha)^2}\int dx\left[e^{i\sqrt{8}\phi_\sigma} + h.c.\right] - \frac{g_f}{(2\pi\alpha)^2}\int dx\left[e^{i\sqrt{8}\theta_\sigma} + h.c.\right]$$

* On the other hand we can write also the SDW and CDW operators using the Klein factors. For example (we use expressions of p. 42, 43 and 35)

$$O^x_{SDW} = \frac{e^{-2ik_Fx}}{2\pi\alpha}\, e^{i\sqrt{2}\phi_\rho}\left[U^+_{R\uparrow}U_{L\downarrow}\, e^{-i\sqrt{2}\theta_\sigma} + U^+_{R\downarrow}U_{L\uparrow}\, e^{+i\sqrt{2}\theta_\sigma}\right]$$

$$O^z_{SDW} = \frac{e^{-2ik_Fx}}{2\pi\alpha}\, e^{i\sqrt{2}\phi_\rho}\left[U^+_{R\uparrow}U_{L\uparrow}\, e^{i\sqrt{2}\phi_\sigma} - U^+_{R\downarrow}U_{L\downarrow}\, e^{-i\sqrt{2}\phi_\sigma}\right]$$

which should be compared with the expressions of p. 42 and 43:

$$O^x_{SDW} = \frac{e^{-2ik_Fx}}{2\pi\alpha}\, e^{i\sqrt{2}\phi_\rho}\left[e^{-i\sqrt{2}\theta_\sigma} + e^{i\sqrt{2}\theta_\sigma}\right]$$

$$O^z_{SDW} = \frac{e^{-2ik_Fx}}{2\pi\alpha}\, e^{i\sqrt{2}\phi_\rho}\left[e^{i\sqrt{2}\phi_\sigma} - e^{-i\sqrt{2}\phi_\sigma}\right]$$

* If the bosonized expressions are correct (I mean those of without Klein factors) then any order in perturbation theory must be identical with and without U factors. Since all the spatial dependence is in the bosonic operators we just should take care of the signs in front of $g_{1\perp}$ and g_f, and this may be fixed by having a look to

first order in perturbation theory (of $e^{-\beta H}$). In addition the signs of $g_{1\perp}$ and g_f may be fixed independently (a change of $\phi_\sigma \to \phi_\sigma + \pi/\sqrt{8}$ or $\theta_\sigma \to \theta_\sigma + \pi/\sqrt{8}$ does the job).

· let's consider first $g_{1\perp}$ and O^z_{SDW}. First without U factors:

Note that

$$e^{-i\sqrt{2}\phi} e^{-\beta H} \overset{1^{st}\text{ order}}{\sim} -g_{1\perp} \int dx'\, e^{i\sqrt{8}\phi' - i\sqrt{2}\phi} \overset{\text{dominantly}}{\sim} -g_{1\perp} e^{i\sqrt{2}\phi}$$

$$e^{i\sqrt{2}\phi} e^{-\beta H} \sim -g_{1\perp} \int dx'\, e^{-i\sqrt{8}\phi' + i\sqrt{2}\phi} \sim -g_{1\perp} e^{-i\sqrt{2}\phi}$$

Hence

$$O^z_{SDW} e^{-\beta H} \sim -g_{1\perp} \left(e^{i\sqrt{2}\phi} - e^{+i\sqrt{2}\phi}\right)$$

· let's see with U factors

$$O^z_{SDW} e^{-\beta H} \sim -g_{1\perp} \int dx' \left[U^+_{R\uparrow} U_{L\uparrow} e^{i\sqrt{2}\phi} - U^+_{R\downarrow} U_{L\downarrow} e^{-i\sqrt{2}\phi}\right]$$

$$\left[U^+_{R\uparrow} U^+_{L\downarrow} U_{R\downarrow} U_{L\uparrow} e^{i\sqrt{8}\phi'} + U^+_{L\uparrow} U^+_{R\downarrow} U_{L\downarrow} U_{R\uparrow} e^{-i\sqrt{8}\phi'}\right]$$

$$\overset{\text{dominantly}}{\sim} -g_{1\perp} \Big[\underbrace{U^+_{R\uparrow} U_{L\uparrow} U^+_{L\uparrow} U^+_{R\downarrow} U_{L\downarrow} U_{R\uparrow}}_{U^+_{R\downarrow} U_{L\downarrow}} e^{-i\sqrt{2}\phi} - \underbrace{U^+_{R\downarrow} U_{L\downarrow} U^+_{R\uparrow} U^+_{L\downarrow} U_{R\downarrow} U_{L\uparrow}}_{U^+_{R\uparrow} U_{L\uparrow}} e^{+i\sqrt{2}\phi}\Big]$$

Recall that for different species they anticommute and for the same species $UU^+ = U^+U = 1$
Cp. (11)

Note that the sign in front of $g_{1\perp}$ is the same as when we forget the U factors. Note also that $U^+_{R\downarrow} U_{L\downarrow} e^{-i\sqrt{2}\phi}$ comes from the bosonization of $\psi^+_{R\downarrow}\psi_{L\downarrow}$, and the same with $\psi^+_{R\uparrow}\psi_{L\uparrow}$.
Hence, we may forget the U's because this choice (O^z_{SDW} with ϕ_σ and H with $+g_{1\perp}$) <u>respects the proper (anti) commutation relations</u>.

* Note that the same could have been done by choosing $O^z_{SDW} \sim \cos\phi_\sigma$ but $H \sim -g_{1\perp}$. We may choose that but we must keep coherent.

* the same analysis can be done with g_f and O^x_{SDW}.

First without U factors:

$$e^{-i\sqrt{2}\theta_\sigma} e^{-\beta H} \sim g_f \int dx' \, e^{i\sqrt{2}\theta - i\sqrt{2}\theta} \sim g_f e^{i\sqrt{2}\theta}$$

$$e^{i\sqrt{2}\theta_\sigma} e^{-\beta H} \sim g_f e^{-i\sqrt{2}\theta}$$

Hence

$$O^x_{SDW} e^{-\beta H} \sim g_f \left(e^{-i\sqrt{2}\theta} + e^{i\sqrt{2}\theta}\right)$$

* Now with U factors:

$$O^x_{SDW} e^{-\beta H} \sim +g_f \left[U^+_{R\uparrow} U_{L\downarrow} U^+_{L\downarrow} U^+_{R\downarrow} U_{L\uparrow} U_{R\uparrow} e^{-i\sqrt{2}\theta} + U^+_{R\downarrow} U_{L\uparrow} U^+_{R\uparrow} U^+_{L\uparrow} U_{R\downarrow} U_{L\downarrow} e^{i\sqrt{2}\theta}\right]$$

$$= g_f \left[U^+_{R\downarrow} U_{L\uparrow} e^{-i\sqrt{2}\theta} + U^+_{R\uparrow} U_{L\downarrow} e^{i\sqrt{2}\theta}\right]$$

and we get the same sign in front of g_f. Note again that $U^+_{R\downarrow} U_{L\uparrow} e^{-i\sqrt{2}\theta}$ comes from the bosonization of $\psi^+_{R\downarrow}\psi_{L\uparrow}$ and the same for $\psi^+_{R\uparrow}\psi_{L\downarrow}$. Hence we may forget the U's because the choice O^x_{SDW} with $\cos\theta_\sigma$ and H with $-g_f$ respects the proper (anti)commutation rules. Again, the same could be done choosing O^x_{SDW} with $\sin\theta_\sigma$ and H with $+g_f$. What we can't do is to take blindly the g_2-form of p. 114 because then we would get the wrong sign! (Similar arguments explain the choices of p. 115)

* LOGARITHMIC CORRECTIONS OF CORRELATION FUNCTIONS

· So far we have determined the asymptotic behavior of correlation functions just by plugging in the renormalized Luttinger liquid parameters in the calculation performed with a purely quadratic Hamiltonian.

· In principle, one should write the renormalization equations for the correlation functions themselves. We will see that this more rigorous analysis justifies our previous treatment, but it also leads to some logarithmic corrections which were absent before.

· Let's start from the Hamiltonian

$$H = H_0 + \frac{2g_\phi}{(2\pi\alpha)^2}\int dx \cos\sqrt{8}\phi - \frac{2g_\theta}{(2\pi\alpha)^2}\int dx \cos\sqrt{8}\theta$$

(Note: as we have done other times, we use $K = 1 + y_\sigma/2$)

· In p. 112 we have written the RG eqs for this Hamiltonian:

$$dy_\sigma/dl = y_\theta^2 - y_\phi^2$$
$$dy_\phi/dl = -y_\phi y_\sigma$$
$$dy_\theta/dl = y_\theta y_\sigma$$

· Let's consider the correlation functions (always time-ordered):

$$\left.\begin{aligned} R_0 &= 2\langle \cos\sqrt{2}\phi(r_1)\cos\sqrt{2}\phi(r_2)\rangle \\ R_1 &= 2\langle \cos\sqrt{2}\theta(r_1)\cos\sqrt{2}\theta(r_2)\rangle \\ R_2 &= 2\langle \sin\sqrt{2}\theta(r_1)\sin\sqrt{2}\theta(r_2)\rangle \\ R_3 &= 2\langle \sin\sqrt{2}\phi(r_1)\sin\sqrt{2}\phi(r_2)\rangle \end{aligned}\right\}$$

which are the $2k_F$ part of CDW, $SDW_{x,y,z}$ (recall p. 41 – 44)

· For $g_\phi = g_\theta = 0$ the correlation functions are easy to compute as in p. 41 – 44:

$$R_i(r_1-r_2) = e^{-K_i F_1(r_1-r_2)}$$

where $\left.\begin{aligned} K_0 &= K_3 = K \\ K_1 &= K_2 = K^{-1} \\ F_1(r) &\simeq \log(r/\alpha) \\ &\quad\hookrightarrow \text{for } r \gg \alpha \end{aligned}\right\}$

We want to see now what happens when $g_\phi, g_\theta \neq 0$

* The idea is to change the cut-off in the correlation function such that

$$R_i(r_1-r_2,\alpha') = I_i(\alpha',\alpha)\, R_i(r_1-r_2,\alpha)$$

One iterates until $r_1-r_2 \sim \alpha$. Hence finding $I(\cdot)$ we will have access to the original correlation function.

* Since $R_i(r)$ has an intrinsic r-dependence even for $g_\theta = s_\phi = 0$ (as we just mentioned), it's better to work with

$\overline{R}_i(r) = e^{K F_i(r)} R_i(r)$

For $s_\phi = s_\theta = 0 \longrightarrow \overline{R}_i = 1$, and hence $I(\alpha',\alpha) = 1$.

This means that we can compute a perturbative expansion of $I(\cdot\cdot)$ in powers of the interaction constants:

$$\overline{R}_i(r,\alpha e^{l}, g(l)) = I_i(dl, g(l))\, \overline{R}_i(r, \alpha e^{l+dl}, g(l+dl))$$

where the different $g(l)$ fulfill the RG eqs of p. 112, and α denotes the initial short-distance cut-off.

Provided that the couplings are still small $\overline{R}_i(r,r,g(\underbrace{\log(r/\alpha)}_{l_r = \log(r/\alpha)})) \sim \mathcal{O}(1)$

and hence we just iterate until reaching $r \sim \alpha$:

$$\overline{R}_i(r,\alpha,g(0)) = \prod_{l=0}^{l_r} I_i(dl, g(l)) = \exp\left\{ \int_0^{l_r} \log\left[I(dl, g(l))\right] \right\}$$

* Let's evaluate for example R_0. Here I'll not do all the calculation, since the procedure is exactly the same as we did already several times (p. 53, p. 110 for example). As always we expand the interaction term in the action and compute the correlation function. Typically ~~[illegible]~~ the first order term vanishes, here it's not the case:

$$R_0(r_1-r_2) = e^{-KF_1(r_1-r_2)}$$
$$-\frac{2g_\phi}{(2\pi\alpha)^2 u}\int d^2r_3 \langle e^{i\sqrt{2}\phi(r_1)} e^{i\sqrt{2}\phi(r_2)} e^{-i\sqrt{8}\phi(r_3)}\rangle$$
$$+\frac{1}{2}\left[\frac{g_\phi}{4\pi^2 u}\right]^2 \sum_{\epsilon_i=\pm 1}\int\frac{d^2r_3}{\alpha^2}\int\frac{d^2r_4}{\alpha^2}\langle e^{i\sqrt{2}\phi(r_1)} e^{-i\sqrt{2}\phi(r_2)} e^{i\epsilon_3\sqrt{8}\phi(r_3)} e^{-i\epsilon_4\sqrt{8}\phi(r_4)}\rangle_{con}$$
$$+\frac{1}{2}\left[\frac{g_\theta}{4\pi^2 u}\right]^2 \sum_{\epsilon_i=\pm 1}\int\frac{d^2r_3}{\alpha^2}\int\frac{d^2r_4}{\alpha^2}\langle e^{i\sqrt{2}\phi(r_1)} e^{-i\sqrt{2}\phi(r_2)} e^{i\epsilon_3\sqrt{8}\theta(r_3)} e^{-i\epsilon_4\sqrt{8}\theta(r_4)}\rangle_{con.}$$

(Note: $\langle\ldots\rangle_{con}$ means after substracting the terms $\langle\ldots\rangle\langle\ldots\rangle$ coming from the partition function in the denominator.)

- Note that now we have a term in first order which was not there before. This term appears because $\sqrt{2}+\sqrt{2}-\sqrt{8}=0$ and hence the correlation function doesn't vanish (recall p. 25). This relates with the fact (p. 76'') that $\cos\sqrt{2}\phi_1\cos\sqrt{2}\phi_2$ behaves for $r_1\sim r_2$ as $\cos\sqrt{8}\phi$.
- We may easily compute the correlation functions (p. 25) to obtain:

$$\overline{R}_0(r_1-r_2) = 1 - e^{2KF_1(r_1-r_2)}\frac{y_\phi}{2\pi}\int_\alpha\frac{d^2r_3}{\alpha^2}e^{-2KF_1(r_1-r_3)}e^{-2KF_1(r_2-r_3)}$$
$$-\frac{y_\phi^2K^2}{2}F_1(r_1-r_2)\int_\alpha^\infty\frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4K} - \frac{y_\theta^2}{2}F_1(r_1-r_2)\int_\alpha^{+\infty}\frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4/K}$$

(Note: as always in the 2nd order terms we perform the gradient expansion. It's actually the same calculation as in p. 111.
In the previous expression $\int_\alpha$ means excluding a radius α around r_1 and r_2)
- As always we shift the cut-off (one has just to take some care with the linear term). We then get:

$$\bar{R}_0(r_1-r_2) = 1 - y_\phi\, dl + \frac{1}{2}(\kappa^2 y_\phi^2 - y_\theta^2)\, F_1(r_1-r_2)\, dl$$
$$+ y'_\phi \int \frac{d^3r}{\alpha'^2}\, e^{2\kappa F_1'(r_1-r_2)}\, e^{-2\kappa F_1'(r_1-r_3)}\, e^{-2\kappa F_1'(r_2-r_3)}$$
$$+ \frac{y_\phi'^2}{2}\kappa^2 F_1'(r_1-r_2)\int_{\alpha'}^{+\infty}\frac{dr}{\alpha'}\left(\frac{r}{\alpha'}\right)^{3-4\kappa} - \frac{y_\theta'^2}{2}F_1'(r_1-r_2)\int_{\alpha'}^{+\infty}\frac{dr}{\alpha'}\left(\frac{r}{\alpha'}\right)^{3-4/\kappa}$$

where $y'_\phi = y_\phi\left(\frac{\alpha'}{\alpha}\right)^{2-2\kappa}$ (recall p. 58 and p. 112)

$y'_\theta = y_\theta\left(\frac{\alpha'}{\alpha}\right)^{2-2/\kappa}$

(Note: here one takes $\kappa \approx 1$ since $y_\theta y_\phi^2$ would be of third order)

· Re-exponentiating:

$$\bar{R}_0(r,\alpha) = \underbrace{\exp\left[-y_\phi dl + \frac{(y_\phi^2 - y_\theta^2)}{2}\log\left(\frac{r}{\alpha}\right)dl\right]}_{I_0(\alpha'/\alpha,\, y_\phi,\, y_\theta)}\bar{R}_0(r,\alpha')$$

Then

$$\bar{R}_0(r) = \exp\left\{\int_0^{l_r}\left[-y_\phi(l) + \frac{1}{2}(y_\phi^2(l) - y_\theta^2(l))\ln\left(\frac{r}{\alpha(l)}\right)\right]dl\right\}$$

Note that

$$\frac{1}{2}\int_0^{l_r} dl\,(y_\phi^2 - y_\theta^2)\log\left(\frac{r}{\alpha}\right) \overset{\text{p. 112}}{=} -\frac{1}{2}\int_0^{l_r} dl\left(\frac{dy_\sigma}{dl}\right)\log\left(\frac{r}{\alpha}\right) \overset{\text{by parts}}{=}$$

$\frac{d}{dl}\log\frac{r}{\alpha} = -\frac{1}{\alpha}\frac{d\alpha}{dl} = -1$

$$= -\frac{1}{2}\int_0^{l_r}\left[\frac{d}{dl}\left(y_\sigma \log\frac{r}{\alpha}\right) + y_\sigma\right]dl = \frac{y_\sigma}{2}\log\left(\frac{r}{\alpha}\right) - \frac{1}{2}\int_0^{l_r} y_\sigma(l)\, dl$$

Hence

$$R_0(r) = \left(\frac{r}{\alpha}\right)^{-\kappa}\bar{R}_0(r) = \left(\frac{r}{\alpha}\right)^{-1-\frac{y_\sigma}{2}} e^{\int_0^{l_r} dl\left(-y_\phi - \frac{y_\sigma}{2}\right)}\left(\frac{r}{\alpha}\right)^{y_\sigma/2}$$

$$R_0(r) = \frac{\alpha}{r}\, e^{-\int_0^{l_r} dl\, y_\phi}\, e^{-\frac{1}{2}\int_0^{l_r} dl\, y_\sigma}$$

* We may proceed in a similar way with the other correlations R_j to obtain that:

$$\left.\begin{aligned} R_0 &= \frac{\alpha}{r} L_1^{-1} L_2 L_3^{-1} \\ R_1 &= \frac{\alpha}{r} L_1 L_2 L_3 \\ R_2 &= \frac{\alpha}{r} L_1 L_2^{-1} L_3^{-1} \\ R_3 &= \frac{\alpha}{r} L_1^{-1} L_2^{-1} L_3 \end{aligned}\right\} \quad \text{with} \quad \begin{aligned} L_1 &= e^{\frac{1}{2}\int_0^{l_r} dl\, y_\sigma} \\ L_2 &= e^{\frac{1}{2}\int_0^{l_r} dl\, (y_\sigma - y_\phi)} \\ L_3 &= e^{\frac{1}{2}\int_0^{l_r} dl\, (y_\sigma + y_\phi)} \end{aligned}$$

* Let's see the case $g_0 = 0$ ($L_3 = L_2^{-1}$). This amount to the case discussed in p. 61.

Let's consider first the separatrix $y_\sigma = y_\phi$.

On the separatrix (p. 61): $y_\sigma^{(l)} = \frac{y_\sigma}{1+y_\sigma l}$, and then:

$$L_1 = \exp\left\{\frac{1}{2}\int_0^{l_r} dl\, \frac{y_\sigma}{1+y_\sigma l}\right\} = \exp\left\{\frac{1}{2}\ln(1+y_\sigma l_r)\right\} = (1+y_\sigma l_r)^{1/2}$$

and $L_3 = L_1$. Hence:

$$R_0 = \frac{\alpha}{r}(1+y_\sigma l_r)^{-3/2} \xrightarrow[r\to\infty]{} \frac{\alpha}{r}\log^{-3/2}\left[\left(\frac{r}{\alpha}\right)^{y_\sigma}\right]$$

and

$$R_1 = R_2 = R_3 = \frac{\alpha}{r}\log^{+1/2}\left[\left(\frac{r}{\alpha}\right)^{y_\sigma}\right]$$

. All correlation functions decay with the same exponent but the logarithmic corrections enhance $R_{1,2,3}$ over R_0.

For spin-isotropic repulsive interactions ~~[illegible]~~ the system "remembers" that it prefers antiferromagnetic order than a CDW. Note that this is provided by the linear term which was absent in our previous RG treatment. Recall that if $\cos\sqrt{8}\phi$ would order it would have done it for $\phi = \pi/\sqrt{8}$, and hence $\cos\sqrt{2}\phi \sim 0$ and $\sin\sqrt{2}\phi \sim 1$, and hence CDW decays exponentially to zero, whereas SDW goes to a constant.

* On the separatrix, however, the cos term isn't yet relevant, but marginal, and hence ϕ doesn't order. However the system "remembers" that $\cos\sqrt{2}\phi$ isn't the same as $\sin\sqrt{2}\phi$ through the different ~~correlation~~ logarithmic corrections.

* Let's see what happens if the system is below the separatrix in the massless regime $y_\sigma > y_\phi$. We use the expressions of p. 61

$$\left.\begin{aligned} y_\sigma(\ell) &= \frac{A}{\tanh\left[A\ell + \operatorname{atanh}(A/y_\sigma)\right]} \\ y_\phi(\ell) &= \frac{A}{\sinh\left(A\ell + \operatorname{atanh}(A/y_\sigma)\right)} \end{aligned}\right\} \quad \text{with } A^2 = y_\sigma^2 - y^2$$

Then: $L_1 = \exp\left\{\frac{1}{2}\int_0^{\ell_r} \frac{A\,d\ell}{\tanh\left[A\ell + \operatorname{atanh}(A/y_\sigma)\right]}\right\} \overset{b=\operatorname{atanh}\frac{A}{y_\sigma}}{\underset{\downarrow}{=}} e^{\frac{1}{2}\int_0^{A\ell_r} \frac{dy}{\operatorname{th}(y+b)}} = e^{\frac{1}{2}\ln\left[\frac{\operatorname{sh}(A\ell_r+b)}{\operatorname{sh} b}\right]}$

$$= \left[\operatorname{ch} A\ell_r + \frac{\operatorname{sh} A\ell_r}{\operatorname{th} b}\right]^{1/2} = \left[\operatorname{ch} A\ell_r + \frac{y_\sigma}{A}\operatorname{sh} A\ell_r\right]^{1/2}$$

$$L_3 = \exp\left\{\frac{1}{2}\int_0^{A\ell_r} \frac{dy}{\operatorname{sh}(y+b)}\right\} = \exp\left\{\frac{1}{2}\ln\left[\frac{\operatorname{th}\left(\frac{A\ell_r+b}{2}\right)}{\operatorname{th} b/2}\right]\right\} = \left(\frac{\operatorname{th}\left(\frac{A\ell_r+b}{2}\right)}{\operatorname{th}(b/2)}\right)^{1/2}$$

Then $\left.\begin{aligned} R_0 &= \frac{\alpha}{r} L_1^{-1} L_3^{-2} \\ R_3 &= \frac{\alpha}{r} L_1^{-1} L_3^{2} \\ R_{1,2} &= \frac{\alpha}{r} L_1 \end{aligned}\right\}$ Since now $L_3 \neq L_1$ then $R_3 \neq R_{1,2}$

• If $A\ell_r \ll 1 \longrightarrow \operatorname{ch} A\ell_r \simeq 1$, $\operatorname{sh} A\ell_r \simeq 1$, $\operatorname{th} A\ell_r \simeq 1$

and hence: $L_1 \simeq \left(1 + \frac{y_\sigma}{A} A\ell_r\right)^{1/2} = (1 + y_\sigma \ell_r)^{1/2}$

$$L_3^2 \simeq 1 + \frac{A\ell_r}{2}\left[\frac{1 - \operatorname{th}^2(b/2)}{\operatorname{th}(b/2)}\right] \simeq 1 + A\ell_r \frac{1}{\operatorname{th} b} = 1 + y_\sigma \ell_r$$

We recover hence $L_1 = L_3 = (1 + y_\sigma \ell_r)^{1/2}$ the same as on the separatrix $(A=0)$.

* Hence for short enough length the system behaves as if the couplings were isotropic.
* However for a length $\log(r/\alpha) \sim A^{-1}$ there's a crossover from isotropic correlation functions ($R_1 = R_2 = R_3$) to an anisotropic regime. For $A\ell_r \gg 1$

$$L_1(r) \simeq \left(\frac{1}{2}\left(1 + y_\sigma/A\right)\right)^{1/2} e^{A\ell_r/2}$$

$$L_3^2(r) \simeq \frac{1}{\tanh(b/2)}$$

and hence $$L_1^{-1} L_3^{-2} \cong \tanh\frac{b}{2}\left[\frac{1}{2}\left(1 + \frac{y_\sigma}{A}\right)\right]^{-1/2}\left(\frac{r}{\alpha}\right)^{-A/2}$$

Then:

$$\left.\begin{array}{l} R_0 \cong \tanh\frac{b}{2}\left[\frac{1}{2}\left(1+\frac{y_\sigma}{A}\right)\right]^{-1/2}\left(\frac{\alpha}{r}\right)^{1+A/2} \\ R_3 \simeq \frac{1}{\tanh b/2}\left[\frac{1}{2}\left(1+\frac{y_\sigma}{A}\right)\right]^{-1/2}\left(\frac{\alpha}{r}\right)^{1+A/2} \end{array}\right\} R_{0,3} \sim \left(\frac{\alpha}{r}\right)^{1+A/2}$$

No logarithmic correction (only the exponent is corrected)

$$R_{1,2} \simeq \left[\frac{1}{2}\left(1+\frac{y_\sigma}{A}\right)\right]^{1/2}\left(\frac{\alpha}{r}\right)^{1-A/2} \quad \Big\} \; R_{1,2} \sim \left(\frac{\alpha}{r}\right)^{1-A/2}$$

* This means that K gets renormalized into

$$K^* = 1 + A/2$$

which is exactly what we got in p. 61. Recall that there the fixed point was $y_\phi^* = 0$, $y_\sigma^* = A$, and hence $K^* = 1 + A/2$.

This result validates our previous RG treatment $\Rightarrow$ irrelevant (even marginal) operators are not able to act on the asymptotic decay of correlations.

* The fact that logarithmic corrections exist up to the anisotropy length scale ($\log(r/\alpha) \sim A^{-1}$) makes that the prefactor of the asymptotic correlation functions is different (it enhances $R_{1,2,3}$ against R_0).