• Spin 1/2 chains

• We will now employ the bosonization techniques to analyze spin 1/2 chains. We will first analyze the simplest case, namely the XXZ Hamiltonian. Later on we will analyze modifications of this model (including frustration), and we will discuss coupled chains.

* The XXZ Hamiltonian

• In the following we consider a 1D spin chain (of spin 1/2). On each site there's a spin $S_i = \sigma_i/2$ (i = site, σ_i = Pauli matrices, $S_i \equiv (S_i^x, S_i^y, S_i^z)$ and $[S_i^\alpha, S_j^\beta] = i\epsilon_{\alpha\beta\gamma} S_j^\gamma \delta_{ij}$).

The spins interact via nearest neighbor exchange (we keep rotation symmetry on the xy plane):

$$H = \sum_i J_{xy} \left(S_{i+1}^x S_i^x + S_{i+1}^y S_i^y \right) + J_z S_{i+1}^z S_i^z$$

→ XXZ HAMILTONIAN

• If $J_{xy} = J_z$ ⟶ Heisenberg Hamiltonian

• If the $\begin{cases} J \text{ are positive} \longrightarrow \text{antiferromagnetic coupling} \\ J \text{ are negative} \longrightarrow \text{ferromagnetic coupling} \end{cases}$

• In a bipartite lattice (i.e. … A B A B A B A B A …) we may introduce the transformation

$$\left.\begin{aligned} S_i^x &\to (-1)^i S_i^x \\ S_i^y &\to (-1)^i S_i^y \\ S_i^z &\to S_i^z \end{aligned}\right\}$$

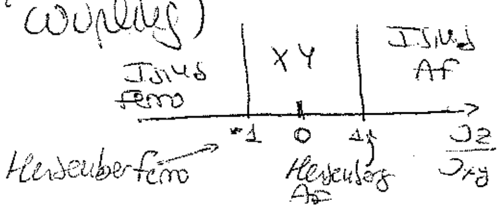
which keeps the proper spin commutation rules. This transformation changes clearly $J_{xy} \to -J_{xy}$ while keeping J_z unchanged.

• We may hence consider $J_{xy} > 0$ without losing generality. We may then study the problem only with J_z/J_{xy}.

- $J_z/J_{xy} = 1$ ⟶ antiferromagnetic isotropic (Heisenberg AF)
- $J_z/J_{xy} = -1$ ⟶ ferromagnetic isotropic (Heisenberg Ferro)
- $J_z/J_{xy} = 0$ ⟶ purely XY (no $S^z S^z$ coupling)

for $J_z > J_{xy}$ ⟶ Ising AF

for $J_z < -J_{xy}$ ⟶ Ising Ferro



- Instead of working with spins we will work with an equivalent model. This may be done in two ways

  - Bosonic systems: we may replace the spin operators by bosonic operators:

$$S^+ \to b^\dagger \qquad S^- \to b \qquad S^z \to b^\dagger b - 1/2$$

This transformation fulfill the proper commutation rules. However, we must impose a hard-core constraint: only one boson per site.

A ~~XXZ~~ system can thus be mapped (actually in any dimension) to a hard-core boson model.

  - Fermionic system: we may replace the spin operators by spinless fermionic operators:

$$S^+ \to c^\dagger \qquad S^- \to c \qquad S^z \to c^\dagger c - 1/2$$

This transformation keeps the proper spin commutation rules when $c, c^\dagger$ fulfill the proper anticommutation rules.

Now we don't need to impose any extra constraint (hard core is now enforced by Pauli exclusion). Now the absence of a fermion is $|\downarrow\rangle$ and the presence $|\uparrow\rangle$.

There's however a problem. Spin operators at different sites commute. This property is not maintained by the previous mapping. We have to change that. This brings us to the Jordan-Wigner transformation

$$\boxed{\begin{aligned} S_i^+ &\to c_i^+ \, e^{i\pi \sum_{j=-\infty}^{i-1} c_j^+ c_j} \\ S_i^z &\to c_i^+ c_i - 1/2 \end{aligned}}$$

Now every S_i^+ has attached a string of operators $e^{i\pi \sum_{j=-\infty}^{i-1} c_j^+ c_j}$, and hence the mapping is clearly non-local.

With this change the mapping now is perfect. The price we had to pay is that (relatively unpleasant) string of operators.

- The string of operators is sometimes a problem, but many times it isn't.

For example:

$$S_{i+1}^+ S_i = c_{i+1}^+ e^{i\pi \sum_{j=-\infty}^{i} c_j^+ c_j} e^{-i\pi \sum_{j=-\infty}^{i-1} c_j^+ c_j} c_i = c_{i+1}^+ e^{i\pi c_i^+ c_i} c_i$$

But $e^{i\pi c_i^+ c_i} c_i = c_i \Rightarrow \left[\begin{array}{l} \text{if } n_i = 0 \to c_i \text{ annihilates} \\ \text{if } n_i = 1 \to c_i|1\rangle = |0\rangle \to e^{i\pi c_i^+ c_i}|0\rangle = |0\rangle \\ \qquad \to e^{i\pi c_i^+ c_i} c_i |1\rangle = |0\rangle = c_i|1\rangle \end{array}\right]$

- Hence $S^+_{i+1} S^-_i = c^+_{i+1} c_i$
- We may then re-write easily the XXZ Hamiltonian in a fermionic way (recall that $(S^+S^- + S^-S^+)/2 = S^x S^x + S^y S^y$)

$$H = \frac{J_{xy}}{2} \sum_i \left[c^+_{i+1} c_i + h.c. \right] + J_z \sum_i (c^+_{i+1} c_{i+1} - 1/2)(c^+_i c_i - 1/2)$$

- Shifting the momentum of the fermions by π (i.e. $c_i \to (-1)^i c_i$)

(Note: that's the same transformation we did in p. (126)):

$$H = -t \sum_i (c^+_{i+1} c_i + h.c.) + V \sum_i \left[c^+_{i+1} c_{i+1} - 1/2 \right] \left[c^+_i c_i - 1/2 \right]$$

with $t = J_{xy}/2$ and $V = J_z$.

The spin chain is hence completely equivalent to a chain of spinless fermions. The fermions hop between neighboring sites with hopping rate t, and experience nearest-neighbor interaction V.

- A magnetic field along z for the spin chain is simply a chemical potential for the fermion:

$$-h \sum_i S^z_i \longrightarrow -h \sum_i (c^+_i c_i - 1/2)$$

For $h=0 \to$ the average magnetization $\langle S_z \rangle = 0$, and $\langle c^+_i c_i \rangle = 1/2$.

The fermionic band is hence half-filled ($k_F = \pi/2$).

On the other hand:

- Totally polarized $\uparrow \longrightarrow$ filled band
- Totally polarized $\downarrow \longrightarrow$ empty band

and actually there's a clear particle-hole symmetry, since a change $c_i \to (-1)^i \tilde{c}^+_i$ changes nothing in the Hamiltonian, except that now $h \to -h$ (particle-hole symmetry $\equiv$ Spin-reversal symmetry $\Rightarrow$ $0 \Leftrightarrow 1$, $\downarrow \Leftrightarrow \uparrow$)

- For $J_z = 0 \Rightarrow H = -t \sum_i (c^+_{i+1} c_i + h.c.) \Rightarrow$ free fermion Hamiltonian

The ground state is then the filled Fermi sea and the excitations are fermionic. We may calculate also some correlations quite easily. For example (fermionic density)

$$\langle S^z(x,\tau) S^z(0,0)\rangle = \langle \rho(x,\tau)\rho(0,0)\rangle$$

but for free fermions we know (p. 29) that the density-density correlations decay as $1/r^2$.

· Unfortunately the string of operators coming from the Jordan-Wigner transformation makes S^+S^- non-trivial in terms of fermionic operators. We will see later how these correlations can be computed.

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* We will now apply the bosonization techniques to the spin-less fermion Hamiltonian:

$$H = -\frac{J_{xy}}{2}\sum_i [c_{i+1}^\dagger c_i + h.c.] + J_z \sum_i (c_{i+1}^\dagger c_{i+1})(c_i^\dagger c_i)$$

we will take the continuum limit and define $S^+(x) = S_i^+/\sqrt{a}$, $S^z(x) = S_i^z/a$, where a is the lattice spacing.

* The kinetic energy term is simple: $c_j = 1/\sqrt{\Omega}\sum_k e^{ikx_j} c_k$

$$H_{kin} = -\frac{J_{xy}}{2}\sum_j (c_{j+1}^\dagger c_j + h.c.) =$$

$$= -\frac{J_{xy}}{2}\sum_{k,k'} e^{-ik'a}\underbrace{\frac{1}{\Omega}\sum_j e^{i(k-k')aj}}_{\delta_{kk'}} c_{k'}^\dagger c_k + h.c.$$

$$= -\frac{J_{xy}}{2}\sum_k 2\cos ka\, c_k^\dagger c_k = \sum_k \underbrace{(-J_{xy}\cos ka)}_{\epsilon_k} c_k^\dagger c_k$$

We get then the standard tight binding result $\epsilon_k = -J_{xy}\cos ka$ for the dispersion of the band. At $k = k_F$ we can expand:

$$\epsilon_k \simeq -J_{xy}\cos k_F a + J_{xy} a \sin k_F a\,(k-k_F)$$

We get then the Fermi velocity $\boxed{v_F = J_{xy}\, a \sin k_F a}$

Hence apart from a constant:

$$H_{kin} = \sum_k (k-k_F) v_F\, c_k^\dagger c_k$$

* Let's see now <u>the interaction part</u>

$$H_{int} = J_z \sum_j S_{j+1} S_j = a^2 J_z \sum_j \rho(x_j+a)\,\rho(x_j)$$

Recall that (p. 15):

$$\rho(x) = \underbrace{\rho_R(x) + \rho_L(x)}_{q\sim 0 \text{ part}} + \underbrace{\psi_R^\dagger(x)\psi_L(x) + \psi_L^\dagger(x)\psi_R(x)}_{q\sim \pm 2k_F \text{ part}} = \quad \leftarrow \text{p. 13, 14}$$

$$= -\frac{1}{\pi}\nabla\phi(x) + \frac{1}{2\pi\alpha}\left[e^{2ik_F x} e^{-2i\phi(x)} + h.c.\right]$$

- Then passing to the continuum (we keep still the $x+a$ explicitely)

$$H_{int} = a J_z \int dx \left[-\frac{1}{\pi}\nabla\phi(x+a) + \frac{1}{2\pi\alpha}\left[e^{2ik_F(x+a)} e^{-2i\phi(x+a)} + h.c.\right]\right]$$
$$\times \left[-\frac{1}{\pi}\nabla\phi(x) + \frac{1}{2\pi\alpha}\left[e^{2ik_F x} e^{-2i\phi(x)} + h.c.\right]\right]$$

$$= a J_z \int dx \Big\{ \frac{1}{\pi^2}\nabla\phi(x+a)\nabla\phi(x) \quad \leftarrow \text{This varies slowly}$$
$$- \frac{1}{\pi}\nabla\phi(x+a)\frac{1}{2\pi\alpha}\left[e^{2ik_F x} e^{-2i\phi(x)} + h.c.\right]$$
$$- \frac{1}{\pi}\nabla\phi(x)\frac{1}{2\pi\alpha}\left[e^{2ik_F(x+a)} e^{-2i\phi(x+a)} + h.c.\right]$$
$$+ \frac{1}{(2\pi\alpha)^2}\left[e^{2ik_F a} e^{-2i[\phi(x+a)-\phi(x)]} + h.c.\right]$$
$$+ \frac{1}{(2\pi\alpha)^2}\left[e^{2ik_F a} e^{i4k_F x} e^{-2i(\phi(x+a)+\phi(x))} + h.c.\right]$$

(The two middle terms: These terms oscillate fast $\sim e^{\pm 2ik_F x}$ (we will neglect them).)

(The fourth term: ← This term varies ~~oscillate~~ slowly.)

(The last term: This term in principle varies fast ($\sim \pm 4k_F$) but we have to take care at half filling (recall p. 85).)

- Since $\phi(x)$ varies slowly

$$1/\pi^2\, \nabla\phi(x+a)\nabla\phi(x) \simeq \frac{1}{\pi^2}(\nabla\phi(x))^2$$

also

$$e^{-2i[\phi(x+a)-\phi(x)]} \simeq 1 + 2ia\nabla\phi(x) - 2a^2(\nabla\phi(x))^2$$

$$e^{2ik_F a} e^{-2i(\phi(x+a)-\phi(x))} + h.c. = 2\cos 2k_F a - 4a\nabla\phi(x)\sin 2k_F a - 4a^2(\nabla\phi)^2\cos 2k_F a$$

~~struck-through expression~~

This is a constant and can be discarded ($2\cos 2k_F a$)

This vanishes with the hermitian conjugate ($-4a\nabla\phi(x)\sin 2k_F a$)

* Let's have a look to the last term. It evolves as $\pm 4k_F$ and hence we would be tempted to drop it. However, as we already know from p. (85), at half-filling ($k_F = \frac{\pi}{2a}$), $4k_F x_j = 4k_F a j = 4\frac{\pi}{2} j = 2\pi j$ ⟶ and hence $e^{\pm 4 k_F x_j}$ doesn't oscillate, and should be hence retained. We have hence to keep the umklapp term, at half-filling ⟵ Note: that for spin chains, half-filling i.e. $\langle S_z \rangle = 0$, is actually a very common case, and hence we have to take good care of the umklapp term

$$e^{2ik_F a} e^{i4k_F x} e^{-2i(\phi(x+a)+\phi(x))} + h.c.$$

$$= e^{i\pi} e^{-2i[\phi(x+a)+\phi(x))} + h.c. \cong -2\cos 4\phi(x)$$

* Combining all these results and removing constants we get (we take $\alpha = a$)

$$H_{int} = a J_z \int \frac{dx}{\pi^2} [1 - \cos 2k_F a](\nabla\phi(x))^2 - \frac{2aJ_z}{(2\pi\alpha)^2} \cos(4\phi(x))$$

* The umklapp term depends as $\cos[4\phi]$ and not $\cos[\sqrt{8}\phi]$ as for spin-full fermions. This means that the term is less relevant. This is because, as we recall from p. (83), the umklapp term comes from interactions $\psi_R^\dagger \psi_R^\dagger \psi_L \psi_L$, but since now we have spinless fermions we can't have two for ψ_L at the same place. These terms aren't zero because what one has in $\psi_L(x+a)\psi_L(x) \simeq [\psi_L(x) + a\nabla\psi_L(x)]\psi_L(x) = a\nabla\psi_L(x)\psi_L(x)$, and hence the umklapp term has hidden derivatives in it.

* Then the Hamiltonian is of the form:

$$H = \frac{1}{2\pi}\int dx\, v_F[(\nabla\theta)^2 + (\nabla\phi)^2] + \frac{1}{2\pi}\int dx \left(\frac{aJ_z 2}{\pi}\right)(1-\cos 2k_F a)(\nabla\phi)^2 - \int dx \frac{2aJ_z}{(2\pi\alpha)^2}\cos 4\phi$$

$$= \frac{1}{2\pi}\int dx \left\{ v_F(\nabla\theta)^2 + \left[v_F + \frac{2aJ_z}{\pi}(1-\cos 2k_F a)\right](\nabla\phi)^2\right\} - \left[\frac{2(aJ_z)}{(2\pi\alpha)^2}\right]\int dx \cos 4\phi$$

* We may hence define:

$$\begin{cases} uK = v_F = J_{xy} a \sin k_F a \\ \dfrac{u}{K} = v_F\left[1 + \dfrac{2aJ_z}{\pi v_F}(1-\cos 2k_F a)\right] \end{cases} \quad \text{and} \quad g_3 = a J_z$$

· The spin chain is hence given by:

$$H = H_0 - \underbrace{\frac{2 g_3}{(2\pi\alpha)^2}\int dx \cos 4\phi(x)}_{\text{Umklapp}}$$

(H_0: Luttinger Liquid)

(at half filling $k_F a = \pi/2$ and
$$u^* = J_{xy} a \left[1 + \frac{4}{\pi}\frac{J_z}{J_{xy}}\right]^{1/2}$$
$$K = \left[1 + \frac{4 J_z}{\pi J_{xy}}\right]^{-1/2}$$)

(Note: since $g_3 = a J_z$, the perturbative limit is J_z small, i.e. we are close to the XY limit (p. (126))).

* Since we have recovered a Hamiltonian as that of p. (83) we may proceed as for that case (recall our discussion of the Mott-U transition in p. (86)). Recall that the g_3 term is irrelevant for $K > 1/n^2$, where the cos term goes as $\cos[n\sqrt{8}\phi]$. Here $n = \sqrt{2}$, and hence we expect a BKT-transition at a critical $K_c = 1/2$

* Hence for $K > 1/2$ (which means $\frac{J_z}{J_{xy}} < \frac{\pi}{4}$ — This factor $\pi/4$ isn't to trust (see below)) the system flows to a fixed point (with K^*) which is Luttinger-liquid like (gapless). Note that the XY limit ($J_z = 0$) is obviously in the massless phase.

* For $K < 1/2$ ($\frac{J_z}{J_{xy}} > \pi/4$ — although the perturbative result for K and U is not to trust here (see below)) the cosine term is relevant and the excitations of our spin chain develop a gap. Since this phase corresponds to a dominant $J_z > J_{xy}$ then the massive phase corresponds to an Ising phase along z.

* The expressions for U and K are perturbative, but, interestingly, the spin-chain model is an example of exactly solvable model (using Bethe-Ansatz). One can actually find that if one defines:

$$\cos \pi\beta^2 = -\frac{J_z}{J_{xy}}$$

then
$$\begin{cases} K = 1/2\beta^2 \\ u = \dfrac{1}{1-\beta^2}\sin[\pi(1-\beta^2)]\,\dfrac{J_{xy}}{2} \end{cases}$$
These are the exact values of the Luttinger-liquid parameters

- For $J_z/J_{xy} \ll 1$ one recovers the perturbative results.

One finds that:

* $K = 1/2$ (i.e. the BKT point) occurs for $\beta = 1$ and hence for $J_z = J_{xy}$ → i.e. at the isotropic Heisenberg point (p. 126)

  So the gap opens when moving from XY to Ising through the Heisenberg point.

* When approaching the isotropic ferromagnetic point ($J_z \to -J_{xy}$) then $\beta \to 0$ and hence K diverges and u vanishes. Hence, the Ising-ferro phase ($J_z < -J_{xy}$) is not a Luttinger liquid.

(Note: in a ferromagnet the spin wave (magnon) dispersion is $\omega \sim k^2$, i.e. quadratic instead of linear.)

* In order to determine correlation functions later on, we should write the spin operators in the boson language.

  Recall that $S_z = \rho - 1/2$ (p. 127). Hence we get (p. 130)

$$\boxed{S_z(x) = -\frac{1}{\pi}\nabla\phi(x) + \frac{1}{2\pi\alpha}\, 2\cos\left[2\phi(x) - 2k_F x\right]}$$

where of course $x_j = aj$

For halffilling ($\mu = 0$): $k_F = \frac{\pi}{2a}$ → $\cos(2\phi - 2k_F x_j) = (-1)^j \cos(2\phi)$

and hence

$$\boxed{S_z^{\text{half-filling}}(x_j) = -\frac{1}{\pi}\nabla\phi(x_j) + \frac{(-1)^j}{\pi\alpha}\cos 2\phi(x_j)}$$

* The S^+ operator is more complicated due to the Wigner-Jordan string (p. 127)
* let's have a look to the string

$$e^{i\pi\sum_{j<i} c_j^\dagger c_j} \cong e^{i\pi\sum_{j<i}\left[\frac{1}{2} - \frac{1}{\pi}\nabla\phi(x_j)\right]} = e^{i\pi\int_{-\infty}^{x} dy\left[\frac{1}{2} - \frac{1}{\pi}\nabla\phi(y)\right]}$$

(≅: dropping fast oscillating terms; =: continuum)

$$= \left[e^{+i\pi\infty/2}\, e^{i\phi(-\infty)}\right] e^{i\pi x/2}\, e^{-i\phi(x)} \;\underset{\text{half-filling}}{\overset{a=1}{=}}\; e^{ik_F x}\, e^{-i\phi(x)}$$

(the bracketed factor: CONSTANT (IRRELEVANT))

* Hence, the string is very simple in terms of the boson fields.

* Let's calculate S^+:

This comes from the transformation of p. 128

$$S^+(x) = (-1)^x\, c^+(x)\, e^{i\pi\int_{-\infty}^{x} dx\, c^+(x)c(x)} = (-1)^x \left[c_R^+(x) + c_L^+(x)\right] e^{i\pi\int_{-\infty}^{x} dx\, \rho(x)} =$$

↑ p. 14

$$\cong \frac{(-1)^x}{\sqrt{2\pi\alpha}} \left\{ e^{-ik_F x} e^{i[\phi(x)-\theta(x)]} + e^{ik_F x} e^{-i[\phi(x)+\theta(x)]} \right\} e^{ik_F x} e^{-i\phi(x)}$$

$$= \frac{(-1)^x}{\sqrt{2\pi\alpha}} \left[e^{-i\theta(x)} + e^{+i[-2\phi(x)-\theta(x)+2k_F x]} \right]$$

This expression is however unpleasantly asymmetric. This can be solved by considering another alternative form for the string:

$$\frac{1}{2}\left[e^{i\pi\sum_{j<i} c_j^+ c_j} + h.c.\right] \cong \frac{1}{2}\left[e^{ik_F x - i\phi(x)} + h.c.\right]$$

Note that this string gives the same mapping as before between fermions and spins, but it's more symmetric. Then:

$$S^+(x) = \frac{(-1)^x}{\sqrt{2\pi\alpha}} \left[e^{-ik_F x} e^{i(\phi-\theta)} + e^{ik_F x} e^{-i(\phi+\theta)}\right] \frac{1}{2}\left[e^{ik_F x - i\phi} + e^{-ik_F x + i\phi}\right]$$

$k_F = \pi/2$

$$= \frac{e^{-i\theta}}{\sqrt{2\pi\alpha}} \left[(-1)^x + \frac{1}{2}(-1)^x \left[e^{2ik_F x} e^{-2i\phi} + e^{-2ik_F x} e^{2i\phi}\right] \right] \Rightarrow$$

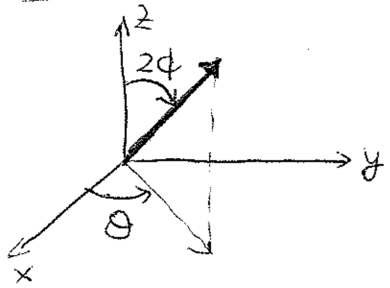
$$\boxed{S^+(x) = \frac{e^{-i\theta(x)}}{\sqrt{2\pi\alpha}} \left[(-1)^x + \cos 2\phi(x)\right]}$$

→ all the nonlocality induced by the string is hidden in the field $\theta(x)$

* Summarizing, for half-filling (absence of magnetization) the spin operators are bosonized as:

$$\boxed{\begin{aligned} S_z(x) &= \frac{-1}{\pi}\nabla\phi(x) + \frac{(-1)^x}{\pi\alpha}\cos(2\phi(x)) \\ S^+(x) &= \frac{e^{-i\theta(x)}}{\sqrt{2\pi\alpha}}\left[(-1)^x + \cos(2\phi(x))\right] \end{aligned}}$$

* This mapping leads to a very simple interpretation of the fields ϕ and θ; they correspond to the polar and azimuthal angle of a classical spin. Because the spin is a quantum object ϕ and θ don't commute.



If θ orders $\rightarrow$ ordering in the xy plane

If ϕ orders $\rightarrow$ " along z.

* We can easy compute the correlation functions:

$$\langle S^z(x,0)\, S^z(0,0)\rangle = C_1 \frac{1}{x^2} + C_2 (-1)^x \left(\frac{1}{x}\right)^{2K}$$

First term: This comes from the slowly varying part ($\nabla\phi$) (recall p. (28)) $q \sim 0$

Second term: This comes from the $\cos(2\phi)$ part (recall p. (28)) $q \sim \pi$ (due to the $(-1)^x$)

$$\langle S^+(x,0)\, S^-(0,0)\rangle = C_3 \left(\frac{1}{x}\right)^{2K+\frac{1}{2K}} + C_4 (-1)^x \left(\frac{1}{x}\right)^{\frac{1}{2K}}$$

First term: This comes from the $e^{-i\theta}\cos 2\phi$ part ($\frac{1}{2K}$ from $e^{-2i\theta}$ and $2K$ from $e^{i2\phi}$) $q \sim 0$

Second term: This comes from the $e^{-i\theta}$ term $q \sim \pi$

where C_j are non-universal constants.

where (recall p (131) and (132)), i.e.

• For the XY case ($J_z = 0$) we have $K = 1$ (recall p(131) and (132)), i.e. we have the free fermion case, and we recover $\langle S_z S_z\rangle \sim 1/x^2$. Note that for S^+S^- the rapidly varying term (that with $(-1)^x$) decays much slower than the slowly varying term ($x^{-1/2}$ vs. $x^{-5/2}$). Hence As a consequence for $K=1 \rightarrow \langle S^+(x)\, S^-(0)\rangle \sim (-1)^x\, x^{-1/2}$, i.e. the staggered magnetization $(-1)^x S^+(x)$ "quasi-orders", i.e. the XY phase has essentially antiferromagnetic "order" in the plane.

* For $K>1$ ($J_z<0$) (attractive fermions) (recall p. 17) there's an enhacement of the 1st term of S^zS^z against the 2nd term, i.e. there's an enhacement of the ferromagnetic term (that with $1/x^2$) vs. the antiferromagnetic term (that with $(-1)^x x^{-2K}$). This was of course to be expected from the form of the XXZ Hamiltonian (p. 126).

* For $K<1$ ($J_z>0$) (repulsive fermions) the opposite is true and the antiferromagnetic part is enhanced. At the isotropic antiferromagnetic point $J_z=J_{xy}$ there's spin rotation symmetry, and one hence expects S_zS_z and S^+S^- to behave identically. This is clearly so for $K=1/2$, in agreement to our discussion of p. 133.

• For $1/2<K<1$ (antiferromagnetic XY regime) one may easily see that the last term of S^+S^- dominates, i.e. the slowest decaying correlation function is the AF part of the XY correlation function.

• For $K=1/2$ (spin isotropic case $J_z=J_{xy}$) the antiferromagnetic part decays as $1/x$ whereas the ferromagnetic one decays as $1/x^2$. But since $J_z=J_{xy}$ is the BKT point the cosine term is marginal and we must consider logarithmic corrections in the spirit of p. 119. We hence expect (recall p. 123):

$$\langle S^\mu(x)S^\mu(x)\rangle = C_1\, 1/x^2 + C_2 (-1)^x \frac{1}{x}\log^{1/2}(x)$$

• For $J_z>J_{xy}$ (Ising AF) the cosine term is relevant, and the spin-less fermions undergo a Mott transition as that studied in p. 86. The field ϕ orders, the system develops a gap, and the XY correlations decay exponentially to zero (since they go with θ). Since H goes as $-\cos 4\phi$, ϕ orders to a value $\phi = 0 + n\pi/2$.

Hence $\langle S_z(x) \rangle = (-1)^x \langle \cos 2\phi \rangle$

$\Longrightarrow$ antiferromagnetic Ising order

$\uparrow \downarrow \uparrow \downarrow \uparrow \ldots \xrightarrow[\text{Fermions}]{} \underset{1}{\bullet}\ \underset{0}{\times}\ \underset{1}{\bullet}\ \underset{0}{\times}\ \underset{1}{\bullet}$ → i.e. a Mott state (with filling 1/2)

- Note that
  - If $\phi = 0 \Rightarrow \ldots \uparrow \downarrow \uparrow \downarrow \uparrow \ldots$
  - If $\phi = \pi/2 \Rightarrow \ldots \downarrow \uparrow \downarrow \uparrow \downarrow \ldots$

  } 2 different ground states where the spins are reversed

The Ising order thus breaks the discrete translational symmetry. Since this is the breaking of a discrete symmetry a true phase transition and order is allowed at $T=0$.

* Effects of a finite magnetic field

* let's consider a magnetic field H along the z direction, which of course will now break the SU(2) symmetry.

* the interaction with the spins of the lattice is given by

$$H_m = - \sum_i g \mu_B H S_i^z \equiv - h \sum_i S_i^z$$

Bosonizing (p. (133)): $H_m = \frac{h}{\pi} \int dx\, \nabla\phi$ ← This has the same form as a chemical potential term (recall the discussion in p. (19) and p. (84)). h is now the chem. pot.

* The magnetic field hence dopes the system compared to half-filling. The problem hence resembles that of p. (84) and we may hence use many of the results found there.

* let's consider the massless region (where the cosine is irrelevant).

The magnetization is

$$m = \langle S_z \rangle = -\frac{1}{\pi} \langle \nabla\phi \rangle$$

let $\tilde{\Phi} = \phi + m\pi x \longrightarrow \langle \nabla\tilde{\Phi} \rangle = 0$

* From our discussion of p. (19) we have $\tilde{\Phi} = \phi + \frac{Kx}{u} h$ and hence

$$\boxed{m = \frac{K}{u\pi} h = \kappa h}$$

↑ compressibility (p. (15))

* We may hence link the magnetization with the Luttinger parameters u and K. However getting the Luttinger liquid parameters is not easy in the presence of a magnetic field.
(Note: due to particle-hole symmetry (i.e. $H \leftrightarrow -H$ symmetry) we just need to consider $m>0$ or $m<0$).

- It's easy to see that $K = 1$ in
  - The pure XY ($J_z = 0$)
  - The limit of nearly empty band (very dilute)

but in general one must obtain the Luttinger-liquid parameters from the numerical solution of Bethe-Ansatz equations.

- When the cosine is relevant, the $J_z=0$ case is gapped (gap Δ). The magnetic field wants to break this gap (recall our discussion of the Mott-δ transition in p. (90)). As for the Mott-δ transition, if $h<\Delta$ no magnetization is induced. When $h>\Delta$ a commensurate-incommensurate phase transition occurs and $m \sim \sqrt{h-\Delta}$

- Two other values of K may be known from the Mott insulator theory. At the BKT point of the Mott-U transition $K = 1/n^2$, i.e. $K = 1/2$, as we already mentioned in p. (132). At the Mott-δ transition, $K = 1/2n^2$, i.e. in our case here $K = 1/4$ (p. (105)).

* When the gap is destroyed, the system is described by a Luttinger-liquid with a shifted $\tilde{\Phi} = \Phi + m\pi x$. This has interesting consequences for the correlation functions. We may inject $\Phi = \tilde{\Phi} - m\pi x$ into the definitions of S_z (p. (133)) and S^+ (p. (134)):

$$S_z(x) = m - \frac{1}{\pi}\nabla\tilde{\Phi} + \frac{(-1)^x}{\pi\alpha}\cos(2\tilde{\Phi} - 2\pi m x)$$

$$S^+(x) = \frac{e^{-i\theta}}{\sqrt{2\pi\alpha}}\left[(-1)^x + \cos(2\tilde{\Phi} - 2\pi m x)\right]$$

Then:

$$\langle S_z(x,0) S_z(0,0)\rangle = m^2 + C_1 \frac{1}{x^2} + C_2 \cos[\pi(1+2m)x]\left(\frac{1}{x}\right)^{2K}$$

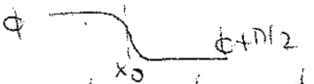
(Note: $(-1)^x \langle \cos(2\tilde{\Phi}(x) - 2m\pi x)\cos 2\tilde{\Phi}(0)\rangle = (-1)^x\, 2\cos 2\pi m x \langle e^{i2(\tilde{\Phi}(x)-\tilde{\Phi}(0))}\rangle$
$= 2\cos(\pi(1+2m)x)\langle e^{i2(\tilde{\Phi}(x)-\tilde{\Phi}(0))}\rangle$)

and:

$$\langle S_+(x,0)\, S_+(0,0)\rangle = C_3 \cos(2\pi m x)\left(\frac{1}{x}\right)^{2K+\frac{1}{2K}} + C_4 \cos\pi x \left(\frac{1}{x}\right)^{\frac{1}{2K}}$$

* Recall that in p. (133) we had terms corresponding to $q\sim 0$ and $q\sim \pi$ excitations. For non-zero magnetic field the $S_z S_z$ correlation has low-energy modes at $q\sim 0$ and $q\sim \pi(1\pm 2m)$, whereas the S^+S^+ correlations has low-energy modes at $q\sim \pm 2\pi m$ and $q\sim\pi$.

* Let's comment finally on <u>the nature of the excitations</u>. We can leave a look to the kinks of the umklapp term $\sim\cos 4\phi$. As discussed in p. (90) the magnetic field (i.e. the holon doping) creates excitations (the holons) that are kinks of the umklapp.



One goes from a minimum of $\cos 4\phi$ to another by $\phi\to\phi+\pi/2$.

We may easily evaluate the spin of such an excitation: (recall p. (90))

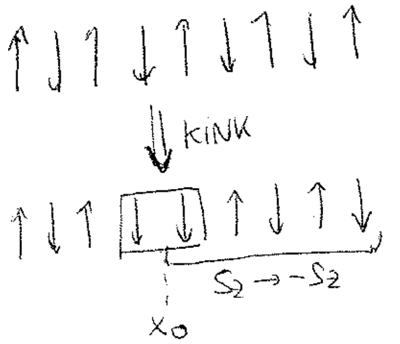
$$\delta S_z = -\frac{1}{\pi}\int_{-\infty}^{x>x_0} dx\, \nabla\phi(x) = -\frac{1}{\pi}\left(\phi(x)-\phi(-\infty)\right) = -\frac{1}{2}$$

We have hence an excitation of spin 1/2. Moreover, after x_0 ϕ remains constant. From the expression of S_z in p. (133)

$$S_z(x) = -\frac{1}{\pi}\nabla\phi + \frac{1}{2\pi\alpha}2\cos(2\phi(x)-2k_F x)$$

we clearly see that after the kink $S_z \longrightarrow -S_z$.

* This means that the excitation is of the following form:



We create a domain wall by reversing all spins after a given point. This is what is called a <u>spinon excitation</u>, and as mentioned before it carries spin 1/2

* Hence the kinks of ϕ correspond to <u>spinons</u>.

* More involved spin chains

* J_1-J_2 chain

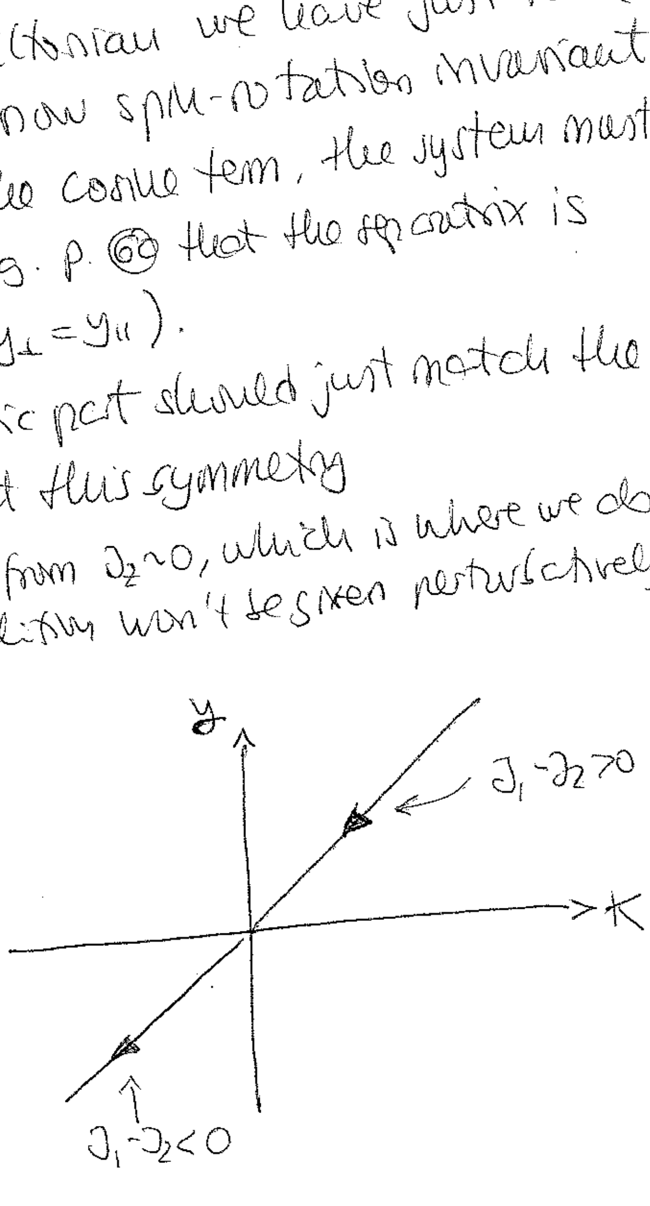
* Let's consider now a more involved lattice. We consider a Heisenberg
exchange (i.e. spin isotropic), but in addition to the nearest neighbor (J_1)
coupling (as we have considered up to now) we have now a next-
nearest neighbor coupling

$$H = J_2 \sum_i S_{i+2} \cdot S_i$$

* This term can be bosonized exactly like the J_1 term.
In the umklapp term in p. (130) we just need to replace $e^{i2k_F a}$
by $e^{i2k_F(2a)}$, and hence at half-filling (no magnetic field, $k_F = \frac{\pi}{2a}$)
$+\cos 4\phi$ instead of $-\cos 4\phi$. Hence in the umklapp term we
have instead of $J_1 \longrightarrow J_1 - J_2$.

* For the quadratic term of the Hamiltonian we have just to be
a little bit careful. The Hamiltonian is now spin-rotation invariant.
This means that when we renormalize the cosine term, the system must
remain on the separatrix (recall from e.g. p. (60) that the separatrix is
characterized by spin-rotation invariance $y_\perp = y_\parallel$).
Hence the coefficient of the quadratic part should just match the
changes of the umklapp term to respect this symmetry.
(Note: Note that the isotropic point is far from $J_2 \sim 0$, which is where we do
perturbation theory, hence the symmetry condition won't be given perturbatively.
We've to be really careful with that!)

* We have then 2 cases
depending whether $J_1 - J_2 > 0$
or viceversa



* If $J_1 - J_2 > 0$ (in fact one can obtain numerically that $J_2 < J_2^{cr}$ with $J_2^{cr}/J_1 \simeq 0.245$)

then the cosine is marginal flowing to zero, i.e. the second neighbor exchange has basically no effect.

* If $J_1 - J_2 < 0$ → the coefficient of the cosine is positive ($y<0$)

The cosine is now relevant (as seen in the RG flow). Hence ϕ orders, but now the coefficient of the cosine is positive, hence ϕ orders into $\phi = \frac{\pi}{4} + \frac{n\pi}{2}$ (i.e. $\cos 4\phi$ goes to its minima).

Recall that S^+ contains $e^{-i\theta}$ (p. 134) and hence the S^+S^- correlation decays exponentially to zero (as it did in p. 136).

But contrary to p. 137, now $\langle \cos 2\phi \rangle \to 0$, i.e. $\langle S_z \rangle \to 0$.

Hence all spin correlations decay to zero exponentially.

The spins in the chain have locked themselves into singlet states due to the frustration induced by the J_2 term.

* Hence for $J_2 > J_2^{c}$ we have a dimerized gapped phase.

* Of course, the presence of a magnetic field may eventually break the gap, very much like in our discussion of p. 137, i.e. we will have a commensurate-incommensurate transition leading to spinon excitations.

* Spin-Peierls Transition

* Let's consider now the situation in which there's a modulation of the coupling:

$$J = J_0 + \delta J \cos Qx$$

We shall consider in particular the case $Q = 2k_F$ and half-filling (no magnetic field) and hence $Q = \pi$, thus:

$$J = J_0 + (-1)^x \delta J$$

· To get an idea of the ground-state of such a system, let's go to strong coupling and assume $\delta J \sim J_0$. In that case the chain is an alternation of strong and weak links

· If for the weak bonds $J_2 = 0$ the ground state is just a product of singlet states around the strong bonds.

· On the strong bond there's a gap ($\sim J_0$) between the singlet and one of the triplets (first excited state). The groundstate is hence stable to a weak J_2.

· Hence for strong coupling ($\delta J \sim J_0$) we expect all spins locked into singlets, and hence all spin-spin correlations decrease exponentially. We expect also a gap ($\sim J_0$) between the ground- and first excited state.

· Let's consider now the case of small dimerization by means of bosonization.

(slowly varying part)

$$S^+_{i+1} S^-_i \xrightarrow[\text{Jordan}]{\text{Wigner}} -c^+_{i+1} c_i = -\left[e^{-ik_F(x+a)} \Psi^+_R(x+a) + e^{ik_F(x+a)} \Psi^+_L(x+a)\right] \cdot \left[e^{ik_F x} \Psi_R(x) + e^{-ik_F x} \Psi_L(x)\right]$$

$$= -\left[e^{-ik_F a} \Psi^+_R(x+a) \Psi_R(x) + e^{ik_F a} \Psi^+_L(x+a) \Psi_L(x)\right]$$

$$+ \left[e^{-ik_F a} e^{-i2k_F x} \Psi^+_R(x+a) \Psi_L(x) + e^{ik_F a} e^{i2k_F x} \Psi^+_L(x+a) \Psi_R(x)\right]$$

* We just consider the $2k_F$ term (the other one enters, as always, in the kinetic quadratic Hamiltonian)

$$[S^+_{i+1} S^-_i]_{2k_F} \cong -\left[e^{-ik_F a} e^{+i2k_F a} \tilde{\psi}^+_2(x)\tilde{\psi}_1(x) + e^{ik_F a} e^{i2k_F x} \tilde{\psi}^+_1(x)\tilde{\psi}_1(x)\right]$$

$$\overset{k_F=\frac{\pi}{2a}}{=} -\left[-i(-1)^x \frac{1}{2\pi\alpha} e^{2i\phi} + i(-1)^x \frac{1}{2\pi\alpha} e^{-2i\phi}\right] = -\frac{(-1)^x}{\pi\alpha}\sin 2\phi$$

* One can then write the Hamiltonian for the spin-Peierls term:

$$H = \frac{\delta J}{2}\sum_i (-1)^i [S^+_{i+1}S^-_i + h.c.] =$$
$$= \frac{-\delta J}{\pi\alpha}\int dx \sin 2\phi$$

By spin rotation symmetry the J_z term should lead to the same physics (and actually one may show that one recovers exactly the same term).

. We have hence once more a sine-Gordon Hamiltonian. Recall that for $\cos(n\sqrt{8}\phi)$ there's a BKT transition into a massive phase at $K_c = 1/n^2$ (recall p. (86)). In this case $n = 1/\sqrt{2}$ and hence $K_c = 2$. Hence the spin-Peierls term is RG-relevant for $K<2$. This means that it's relevant basically unless one approaches the ferromagnetic phase (recall from p. (133) that K diverges in the ferromagnetic phase). This includes the XY point ($K=1$) and the Heisenberg point ($K=1/2$).

. Far from $K=2$ (deep in the massive phase) we may employ the discussion of p. (64) to evaluate the gap Δ.

(Note: in p. (64) we had $\Delta \sim y_0^{1/(2-2K)}$. This is because we had $\cos\sqrt{8}\phi$, and $\langle\cos\sqrt{8}\phi(r)\cos\sqrt{8}\phi(0)\rangle \sim (\alpha/r)^{4K}$. But now we have $\sin 2\phi$ and $\langle\sin 2\phi \sin 2\phi\rangle \sim (\alpha/r)^{2K}$ → hence where we had $2K$ now we have simply K.)

We have hence: $\boxed{\Delta \propto (\delta J)^{1/(2-K)}}$

- For the XY point ($K=1$) $\rightarrow$ $\Delta \propto (\delta J)$

  For the Heisenberg point ($K=1/2$) $\rightarrow$ $\Delta \propto (\delta J)^{2/3}$

the gap is hence much enhanced at the Heisenberg point compared to the XY point.

- The dimerization is hence a very strong effect. We may easily understand it from the fermion picture. The δJ modulation creates a periodic potential for the fermions, and hence in the presence of repulsion the instability towards a CDW is enhanced.

- In the gapped phase, due to the $-\delta J \sin 2\phi$ dependence, the field ϕ orders at

$$\phi = \frac{\pi}{4} + \pi n \qquad \delta J > 0$$

$$\phi = -\frac{\pi}{4} + \pi n \qquad \delta J < 0$$

Since ϕ orders, the S^+S^- decay exponentially to zero (S^+ depends on $e^{i\theta}$). For $S^z \sim \langle \cos 2\phi \rangle = 0$, hence all S^z correlations also decay exponentially.

* Since a shift $\phi \rightarrow \phi + \pi/2$ corresponds to a shift of the spin modulation by one lattice spacing (recall p. (133)), then depending on the sign of δJ the ground state shift by one lattice site the location of the spins. We thus recover the previous interpretation of the ground state in terms of a locking of the spins in singlets on the strongest bonds.

* The weak and strong coupling thus give the same physics for the spin-Peierls state and are smoothly connected.

* Finally let's briefly mention what happens in the presence of a magnetic field. We may proceed like in our discussion of the Mott-δ transition (p. (90)) or our discussion of p. (106). As for those cases, if the magnetic field is larger than the gap, then it destroys the commensurate state into a incommensurate one (C-IC transition).

Recall that for a sine-Gordon with $\cos(n\sqrt{8}\phi)$ one has a universal $K^* = 1/2n^2$ at the transition. For the Ising-AF case we had $\cos(4\phi)$ in the umklapp, and hence $n=\sqrt{2}$, hence $K^* = 1/4$ (as we mentioned in p. (138)). Here, on the contrary we have the spin-Peierls term $\sim \sin 2\phi$, i.e. $n = 1/\sqrt{2}$, and hence $K^* = 1$ at the Mott-δ transition.

As in p. (139) we may have also a look in the excitations. One goes from a minimum of $-\sin 2\phi$ to another minimum by a $\delta\phi = \pi$ kink.

$$\delta S_z = -\frac{1}{\pi}(\phi(x_0) - \phi(-\infty)) = -\frac{\pi}{\pi} = -1$$

The excitations carry hence spin 1 and not spin 1/2 as for the Ising-AF (or the frustrated J_1-J_2 massive phase)

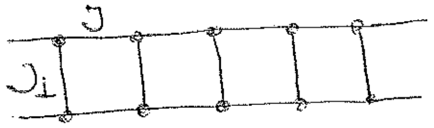
* The excitation are here triplets ($S=1$) and not spinons ($s = 1/2$).

* Coupled Spin Chains

* We will consider now the case of coupled chains. We will first analyze the case of spin ladders, and then we will consider an infinite number of chains.

* Spin ladders

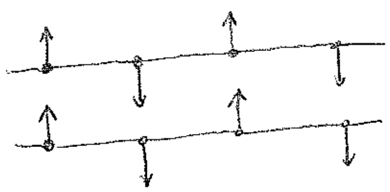
* Let's consider a two-leg ladder of the form. For simplicity we consider isotropic couplings:



  * Intrachain $\to J \to$ (we consider it antiferromagnetic)
  * Interchain $\to J_\perp$

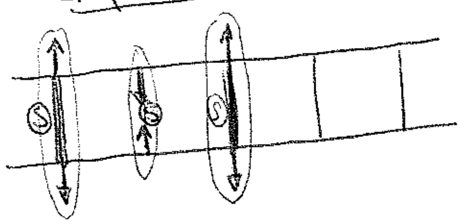
* Let's consider first the case of large $J_\perp$. In this case we cannot use bosonization (if $J_\perp > J$ the coupling is larger than the 1D bandwidth and we can't linearize anymore). However we can get a good understanding of the physics rather easily:

  * If $J_\perp$ is ferromagnetic: the spins in a rung are locked in a maximally polarized state. The two-leg ladder is hence equivalent to a $S=1$ antiferromagnetic spin chain (for n-legs $\to S = n/2$).



(Note: This is interesting because $S = n/2$ chains follow the so-called Haldane conjecture, according to which n even shows drastically different properties compared to n odd.)

  * If $J_\perp$ is antiferromagnetic: the spins in a rung are locked in a singlet state. These singlets have a gap ($\sim J_\perp$) against triplet excitations. The (weaker) intrachain coupling J can't change this ground state. Hence the ground-state of the ladder is a series of singlets on each rung and totally gapped.



(Note: in a 3-leg ladder, one can't make a singlet since one spin remains unpaired. Each rung can hence be replace by a spin-1/2, and the ladder is equivalent to a spin-1/2 chain, and thus gapless. One see hence a crucial difference between odd and even number of legs!)

• Let's consider now the case of weak $J_\perp$, such that we can still do bosonization. Each single chain Hamiltonian can be expressed in Fermi operators using Wigner-Jordan (taking care that the "Fermi" operators commute between chains to keep the spin commutation relations). Actually we may directly use the bosonization expressions for S_z and S^+ we introduced in p. (133) and (134).

Let's consider the two-leg ladder:

$$H = \sum_{\alpha=1,2} H_\alpha^0 + \underbrace{\frac{J_\perp^{xy}}{2} \sum_j \left[S_{j1}^+ S_{j2}^- + S_{j1}^- S_{j2}^+\right] + J_\perp^z \sum_j S_{j1}^z S_{j2}^z}_{H_{inter} \equiv \text{inter-leg interaction}}$$

We may bosonize H_α^0 (intra-leg Hamiltonians) as we did in p. (132), so we just need to have a look to the extra inter-leg interactions to the bosonization expressions (no magnetic field)

We employ the bosonization expressions

$$S^+(x) = \frac{e^{-i\theta(x)}}{\sqrt{2\pi a}}\left[(-1)^x + \cos 2\phi(x)\right] \qquad (\text{p. } 134)$$

$$S^z(x) = -\frac{1}{\pi}\partial_x\phi + \frac{(-1)^x}{\pi a}\cos 2\phi(x) \qquad (\text{p. } 133)$$

and recall that for the passing to the continuum we have (p. (129)) $S_i^+ \Rightarrow \sqrt{a}\, S^+(x)$, $S_i^z \Rightarrow a\, S^z(x)$ and as always $\sum_i \to \frac{1}{a}\int dx$

Then:

$$H_{inter} = \frac{J_\perp^{xy}}{2}\int dx \left[S_1^+(x) S_2^-(x) + S_1^-(x) S_2^+(x)\right] + J_\perp^z a \int dx\, S_1^z(x) S_2^z(x)$$

$$= \frac{J_\perp^{xy}}{}\int dx \left\{ \frac{\cos[\theta_1(x)-\theta_2(x)]}{2\pi a}\begin{bmatrix} 1+\cos 2\phi_1(x)\cos 2\phi_2(x) \\ +(-1)^x(\cos 2\phi_1(x) + \cos 2\phi_2(x))\end{bmatrix}\right\}$$

$$+ J_\perp^z a \int dx \left\{ \frac{1}{\pi^2}\partial_x\phi_1 \partial_x\phi_2 + \frac{1}{\pi^2 a^2}\cos 2\phi_1 \cos 2\phi_2 + \frac{(-1)^{x+1}}{\pi^2 a}\left(\partial_x\phi_1 \cos 2\phi_2 + \partial_x\phi_2 \cos 2\phi_1\right)\right\}$$

$$= \frac{J_\perp^{xy}}{2\pi a}\int dx \cos[\theta_1 - \theta_2] + \frac{J_\perp^z a}{\pi^2}\int dx\, \partial_x\phi_1 \partial_x\phi_2 +$$

$$+ \frac{J_\perp^z a}{\pi^2 a^2}\int \cos 2\phi_1 \cos 2\phi_2 \, dx$$

(↑ only the most relevant terms)

(<u>Note</u>: The other terms are either $\langle \ldots \rangle = 0$ or they decay much faster (as it's the case of $\cos 2\phi_1 \cos 2\phi_2 \cos(\theta_1-\theta_2)$)).

* Then, we may re-write:

$$H_{inter} = \int dx \left\{ \frac{2g_1}{(2\pi a)^2} \cos(\theta_1-\theta_2) + \frac{2g_2}{(2\pi a)^2} \cos 2(\phi_1-\phi_2) + \frac{2g_3}{(2\pi a)^2} \cos 2(\phi_1+\phi_2) \right\}$$

$$+ J_\perp^z a \int dx \frac{\partial_x \phi_1 \, \partial_x \phi_2}{\pi^2}$$

where $g_1 = \pi J_\perp^{xy} a$

$g_2 = g_3 = J_\perp^z a$

* We will now rewrite the whole Hamiltonian in terms of $\phi_{\substack{s\\a}}(x) = \frac{1}{\sqrt{2}}(\phi_1(x) \pm \phi_2(x))$ and $\theta_{\substack{s\\a}}(x) = \frac{1}{\sqrt{2}}(\theta_1(x) \pm \theta_2(x))$

* let's re-write the quadratic part $H_1^0 + H_2^0 + \frac{J_\perp^z a}{\pi^2} \int dx \, \partial_x \phi_1 \, \partial_x \phi_2$.

Recall that:

$$H_\alpha^0 = \frac{1}{2\pi} \int dx \left[uK (\partial_x \theta_\alpha)^2 + \frac{u}{K} (\partial_x \phi_\alpha)^2 \right] - \frac{2 g_{umklapp}}{(2\pi a)^2} \int dx \cos 4\phi_\alpha$$

Note that <u>the umklapp term goes with $\cos 4\phi$ which is always less relevant than the terms from Hinter and hence it may be dropped</u>. Hence

$$\sum_{\alpha=1,2} H_\alpha^0 + \frac{J_\perp^z a}{\pi^2} \int dx \, \partial_x \phi_1 \, \partial_x \phi_2 = \frac{1}{2\pi} \int dx \left\{ uK (\partial_x \theta_s)^2 + \frac{u}{K} (\partial_x \phi_s)^2 \right\}$$

$$+ \frac{1}{2\pi} \int dx \left\{ uK (\partial_x \theta_a)^2 + \frac{u}{K} (\partial_x \phi_a)^2 \right\} + \frac{J_\perp^z a}{2\pi^2} \int dx \left[(\partial_x \phi_s)^2 - (\partial_x \phi_a)^2 \right]$$

$$= \frac{1}{2\pi} \int dx \left\{ u_s K_s (\partial_x \theta_s)^2 + \frac{u_s}{K_s} (\partial_x \phi_s)^2 \right\} + \frac{1}{2\pi} \int dx \left\{ u_a K_a (\partial_x \theta_a)^2 + \frac{u_a}{K_a} (\partial_x \phi_a)^2 \right\}$$

where: $K_{\substack{s\\a}} = \frac{K}{\sqrt{1 \pm \frac{K J_\perp^z a}{\pi u}}} \simeq K \left[1 \mp \frac{K J_\perp^z a}{2\pi u} \right]$

$$u_{\substack{s\\a}} = \frac{uK}{K_{\substack{s\\a}}} \simeq u \left[1 \pm \frac{K J_\perp^z a}{2\pi u} \right]$$

* We can hence separate the Hamiltonian as $H = H_s + H_a$, where we have:

* Symmetric part

$$H_s = \int \frac{dx}{2\pi} \left[u_s K_s (\partial_x \theta_s)^2 + \frac{u_s}{K_s} (\partial_x \phi_s)^2 \right] + \frac{2g_2}{(2\pi a)^2} \int dx \cos \sqrt{8}\, \phi_s$$

* Antisymmetric part

$$H_a = \int \frac{dx}{2\pi} \left[u_a K_a (\partial_x \theta_a)^2 + \frac{u_a}{K_a} (\partial_x \phi_a)^2 \right] + \frac{2g_3}{(2\pi a)^2} \int dx \cos \sqrt{8}\, \phi_a$$

$$+ \frac{2g_1}{(2\pi a)^2} \int dx \cos \sqrt{2}\, \theta_a$$

* Let's consider the isotropic (Heisenberg) case. Then $K = 1/2$ (p. 133), and hence $K_s, K_a \approx 1/2$.

• Note that H_s is a sine-Gordon Hamiltonian (with $\cos n\sqrt{8}$ with $n=1$) and hence one has a massive phase for $K_s < 1/n^2 = 1$ (p. 86). Since $K_s \approx 1/2$ this means that the symmetric sector is massive.

• The antisymmetric sector is more complicated, since it has a competition between $\cos\sqrt{8}\,\phi_a$ and $\cos\sqrt{2}\,\theta_a$

(Note: This resembles what we found in p. 110 for the case of magnetic anisotropies where we had both $\cos\sqrt{8}\,\phi$, $\cos\sqrt{8}\,\theta$ anisotropies)

Contrary to the case of magnetic anisotropies where we had $\cos\sqrt{8}\phi$ and $\cos\sqrt{8}\theta$, here we have $\cos\sqrt{8}\phi$ and $\cos\sqrt{2}\theta$. Note that $\cos n\sqrt{8}\,\phi$ is relevant for $K < \frac{1}{n^2} = 1$ ($n=1$) whereas $\cos n\sqrt{8}\,\theta$ is relevant for $\frac{1}{K} < 1/n^2 \overset{n=1/2}{=} \frac{1}{4} \Rightarrow K > 1/4$

(Note: recall that for the expressions with θ we have always $1/K$ instead of K)

Hence both terms are relevant for $K_a \approx 1/2$. The most relevant operator under renormalization is the first to attain the strong coupling regime (acquires a mean value) whereas

• For $K_a < 1/2 \Rightarrow \phi_a$ is ordered (acquires a mean value) whereas θ_a has exponentially decaying correlations

• For $K_a > 1/2 \Rightarrow$ viceversa

- Hence both the symmetric and antisymmetric sectors are massive, all spin correlations decay exponentially, and for AF $J_\perp$ one recovers that all spins are locked into a singlet state
(Note: $\langle \cos\sqrt{8}\phi_a \rangle \to$ minimal $\to \phi_a = \pi/\sqrt{8} \to \phi_1 - \phi_2 = \pi/2 \to$ ↑↓ ↓↑)
($g_3 > 0$)

One thus recovers the result found for the strong-coupling (p. 146).

(Note: One may also recover that even-leg ladders are gapped, whereas odd-leg ladders are gapless ⟶ Haldane's conjecture).

- Finally note that a magnetic field couples to the total spin, and hence to the symmetric sector:

$$H_{mag} = h \frac{\sqrt{2}}{\pi} \int dx\, \nabla\phi_s(x) \qquad (h \equiv g\mu_B h)$$

One will have then a Mott-δ transition but only in the symmetric sector, whereas the antisymmetric sector remains gapped! This means that some correlation functions in the ladder still decay exponentially even in the presence of the magnetic field!

- Infinite number of chains

- We have seen what happens in a ladder, let's see now what happens in the case of an infinite number of coupled spin chains. We restrict our discussion to the case without magnetic field and with isotropic spin couplings. We consider a 3D lattice of chains.

- The interchange coupling is:

$$H_\perp = J_\perp \sum_{\langle \mu\nu \rangle} \int dx\, S_\mu(x) \cdot S_\nu(x)$$

At low T one expects the system to order. Since the antiferromagnetic order is the slowest-decaying correlation in 1D, one expects the 3D coupling to stabilize this order. One can thus try to treat the interchain coupling in mean-field:

$S_\mu = \langle S_\mu \rangle + \delta S_\mu$

$$\Longrightarrow S_\mu S_\nu = [\langle S_\mu \rangle + \delta S_\mu][\langle S_\nu \rangle + \delta S_\nu]$$

$$\simeq \langle S_\mu \rangle \langle S_\nu \rangle + \langle S_\mu \rangle \delta S_\nu + \langle S_\nu \rangle \delta S_\mu$$

$$= \langle S_\mu \rangle \langle S_\mu \rangle + \langle S_\mu \rangle [S_\nu - \langle S_\mu \rangle] + \langle S_\nu \rangle [S_\mu - \langle S_\mu \rangle]$$

$$= \langle S_\mu \rangle S_\nu + \langle S_\nu \rangle S_\mu - \langle S_\mu \rangle \langle S_\nu \rangle$$

For a staggered field (which is what we expect for the AF order) $\langle S_\mu(x) \rangle = -\langle S_\nu(x) \rangle$ for nearest neighbors

Then:

$$H_\perp \simeq J_\perp \sum_{\langle \mu,\nu \rangle} \int dx \, [S_\mu(x) \langle S_\mu(x) \rangle + \langle S_\mu(x) \rangle \cdot S_\nu(x) - \langle S_\mu(x) \rangle \cdot \langle S_\nu(x) \rangle$$

The Hamiltonian of each chain is thus affected by the mean-field interaction with the neighbors:

$$H_\mu = H_\mu^0 + J_\perp \sum_\nu S_\mu(x) \cdot \langle S_\nu(x) \rangle \qquad \nu \equiv \text{neighbors of } \mu$$

Let's assume that the order takes place along z: $\langle S_\nu^z \rangle \neq 0$, $\langle S_\nu^{x,y} \rangle = 0$

$$H_\mu = H_\mu^0 + J_\perp \sum_\nu S_\mu^z(x) \langle S_\nu^z(x) \rangle$$

The average $\langle S_\nu^z(x) \rangle$ is a staggered function, which will act as an effective staggered magnetic field:

$$h = Z J_\perp \langle S_\nu^z(x) \rangle = -Z J_\perp \langle S_\mu^z(x) \rangle \qquad \text{(self-consistency condition)}$$

Z ↓ coordination number (number of nearest neighbors)

Then:

$$\boxed{H_\mu = H_\mu^0 + h \int dx \, (-1)^x S^z(x)}$$

← we have hence reduced the problem to a spin chain in a staggered field

If the interchain coupling is small compared to the intra-chain one we can use bosonization.

Recall that: $S^z(x) = -\frac{1}{\pi} \nabla \phi + \frac{(-1)^x}{\pi \alpha} \cos 2\phi(x)$. For a non-staggered field (for no mag. field) we keep $\nabla \phi$ (p. 137) but for the staggered field we must keep the cosine part:

$$H_\mu = H_\mu^0 + \frac{h}{\pi\alpha}\int dx \cos 2\phi(x)$$

we recover hence, once more, a sine-Gordon Hamiltonian.

· The <u>self-consistency</u> demands:

$$h = -z J_\perp \langle S_\mu^z(x)\rangle = -\frac{z J_\perp}{\pi\alpha}\langle \cos 2\phi(x)\rangle$$

For an isotropic coupling ($K = 1/2$) ⟶ the cosine is relevant ($K_c = 2$), the average of the cosine isn't zero, and thus <u>there's a solution of the self-consistency equation.</u>

Let's have a look to the consistency equation. Since h is supposed to be small (small $J_\perp$) we may expand the correlation:

$$\langle \cos 2\phi_{(x_0,\tau_0)}\rangle = \frac{1}{Z}\int D\phi\, e^{-S_0} e^{-\frac{h}{\pi\alpha}\int^\beta d\tau\int dx \cos 2\phi(x,\tau)} \cos 2\phi(x_0,\tau_0)$$

$$Z = \int D\phi\, e^{-S_0} e^{-\frac{h}{\pi\alpha}\int^\beta d\tau \int dx \cos 2\phi(x,\tau)}$$

$$\cong Z_0\left[1 - \frac{h}{\pi\alpha}\int^\beta d\tau \int dx \cancel{\langle \cos 2\phi(x,\tau)\rangle_0}^{\,0}\right]$$

Hence

$$\langle \cos 2\phi(x_0,\tau_0)\rangle \simeq \cancel{\langle \cos 2\phi(x_0,\tau_0)\rangle_0}^{\,0} - \frac{h}{\pi\alpha}\int^\beta d\tau\int dx \langle \cos 2\phi(x_0,\tau_0)\cos 2\phi(x,\tau)\rangle$$

Hence:

$$h = \frac{z J_\perp h}{(\pi\alpha)^2}\int^\beta d\tau\int dx \langle \cos 2\phi(x_0,\tau_0)\cos 2\phi(x,\tau)\rangle_0$$

We have a non-trivial solution $h \neq 0$ at the temperature β_c:

$$\boxed{\frac{1}{zJ_\perp} = \frac{1}{(\pi\alpha)^2}\int^{\beta_c} d\tau\int dx \langle \cos 2\phi(x_0,\tau_0)\cos 2\phi(x,\tau)\rangle_0}$$

The transition occurs then when the inverse transverse coupling $\left(\frac{1}{zJ_\perp}\right)$ equals the 1D susceptibility (right side).
↑ spin-spin

* If the 1D susceptibility diverges, there's always a temperature at which the system orders 3D.

On the other hand, if the susceptibility doesn't diverge at $T=0$ (i.e. at $\beta=\infty$) the system might not order if $J_\perp$ is too weak.

Since $\langle \cos 2\phi_0 \cos 2\phi \rangle_0 = \frac{1}{2}\langle e^{+i2(\phi_0-\phi)}\rangle = \frac{1}{2} e^{-2\langle(\phi_0-\phi)^2\rangle}$

$$= \frac{1}{2} e^{-2\frac{K}{2}\ln\left(\frac{r}{\alpha}\right)^2} = \frac{1}{2}\left(\frac{\alpha}{r}\right)^{2K}$$

Then: $\int^{\beta_c} d\tau \int dx \langle \cos 2\phi_0 \cos 2\phi \rangle \propto \beta_c^{2-2K} = \frac{1}{(T_c)^{2-2K}}$

Hence $\frac{1}{(T_c)^{2-2K}} \propto \frac{1}{J_\perp} \longrightarrow \boxed{T_c \propto J_\perp^{\frac{1}{2-2K}}}$

The critical T is thus strongly renormalized by the 1D fluctuations. For very weak $J_\perp$ we <u>can't do</u> mean-field, because we would forget that 1D quantum fluctuations can strongly disorder the system, grossly overestimating the critical temperature (for example for $K\to 1$ T_c would diverge).