Circular restricted three-body problem

Although a problem with two gravitationally interacting point masses is analytically solvable, a system containing three gravitationally interacting point masses is in general not analytically solvable. A rather simplified three-body problem occurs when one of the masses is much smaller than the other two: $m_3 \ll m_1, m_2$. This may be e.g. the case of a comet under the gravitational influence of Jupiter and the Sun. In that case, whereas m_3 is affected by the other two masses, the big masses are not affected by m_3. We may hence solve separately the two-body problem for masses m_1 and m_2, and then study the evolution of m_3 in the gravitational field of the other two masses. As a further simplification we will assume that the two big masses execute circular orbits about their center of mass. So formulated, this is the so-called *circular restricted three-body problem.*
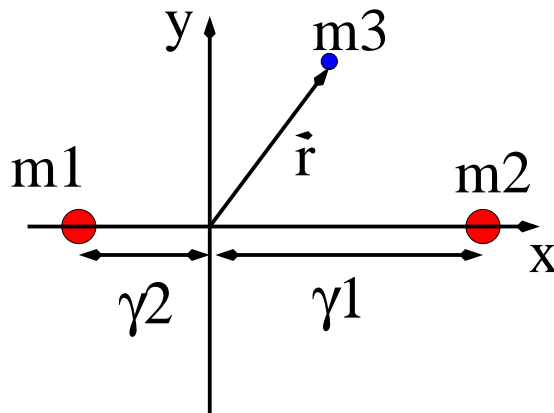
Assuming for simplicity the (constant) relative diatance $R = 1$ as the unit of length, and $G(m_1 + m_2) = 1$ as the unit of mass, we may define $\gamma_1 = Gm_1$ and $\gamma_2 = Gm_2 = 1 - \gamma_1$, and the positions of $m_{1,2}$ will be hence: $\vec{r}_1 = -\gamma_2(\cos t, \sin t, 0)$, and $\vec{r}_2 = \gamma_1(\cos t, \sin t, 0)$.

The third mass has a position $\vec{r} = (x, y, z)$. The equations of motion for $\vec{r}(t)$ will be given by the gravitational force exerted by the other two masses. It is convinient to use the non-inertial reference frame rotating with the big masses (with rotation frequency $\vec{\omega} = \omega \vec{e}_z$, with $\omega = 1$). Recall that in the non-inertial frame we need to add the effect of both the Coriolis and the centrifugal forces:

$$m_3 \ddot{\vec{r}} = \vec{F}_{13}(\vec{r}) + \vec{F}_{23}(\vec{r}) - 2m_3 \vec{\omega} \times \dot{\vec{r}} - m_3 \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

where, for $j = 1, 2$, $\vec{F}_{j3}(\vec{r}) = -m_3 \gamma_j \frac{(\vec{r} - \vec{r}_j)}{|\vec{r} - \vec{r}_j|^3}$, with $\vec{r}_1 = -\gamma_2 \vec{e}_x$ and $\vec{r}_2 = \gamma_1 \vec{e}_x$ the position of the two big masses in the rotating frame (see the figure).

We have hence three coupled second-order differential equations for x, y and z, which demand six initial conditions given by $\vec{r}(0)$ and $\dot{\vec{r}}(0)$.

- Write a program to solve these equations (see the additional notes at the end of the exercise).

Once you have written the code, you may start investigating the dynamics of m_3.

- Check that:

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x, y, z) \tag{1}$$

  is a constant of motion, with $U(x, y, z) = -\frac{\gamma_1}{|\vec{r} - \vec{r}_1|} - \frac{\gamma_2}{|\vec{r} - \vec{r}_2|} - \frac{1}{2}(x^2 + y^2)$ the effective potential resulting from the sum of the gravitational and the centrifugal potentials.

In order to analyse the dynamics it is convenient to visualize the trajectories. In Mathematica it is particularly useful for this purpose the command ListAnimate (see the additional notes at the end of the exercise). If you use C or FORTRAN codes you may visualize the data files using e.g. gnuplot if you work under Linux.

- You may have a look on how sensitive are the orbits of m_3 with respect to the initial conditions. Consider for example $m_1 = m_2$ and two very close initial conditions for the position of m_3. What happens with the trajectories after a sufficiently long time? Explore it for different initial conditions.

Another important question is whether the particle m_3 can scape from the gravitational attraction of the big masses. Consider the case $m_1 \gg m_2 \gg m_3$. This may be the case of say the Sun (m_1), Jupiter (m_2), and a comet (m_3). .

- Let us first consider a very small m_2, e.g. $m_2 = 0.0001 m_1$. Consider an initial condition $\vec{r}_0 = (x_0, y_0, 0)$. Investigate the trajectories for different values of $r_0 = |\vec{r}_0|$. You should see that there is a critical value $(r_0)_{cr}$ such that for $r_0 > (r_0)_{cr}$ the comet scapes to infinity? Determine numerically that value.

- What happens if $r_0 < (r_0)_{cr}$?

- Challenge: Can you guess actually why the system behaves like this? Hint: You should have a look to the two-body problem $m_1 \gg m_3$, and solve the Kepler problem in the rotating frame. By doing that you may even get an analytical form for $(r_0)_{cr}$ that you may compare with the numerically found.

- Compare the case of a negligible m_2 to a case in which m_2 is non-negligible, but still $m_2 \ll m_1$, e.g. $m_2 = 0.01$. You will see that the effect of m_2 is especially remarkable if r_0 is in the vicinity of $(r_0)_{cr}$. What may happen in that case?

Finally, let us have a look to some peculiar points, known as Lagrange points, at which (in the rotating frame) $\dot{\vec{r}} = \ddot{\vec{r}} = 0$. There are five of them, traditionally denoted as $L_{j=1,...,5}$. Let us assume $\gamma_2 \ll \gamma_1$, which may be e.g. the case of the Sun and Earth.

a) L_1 is located at $y = z = 0$, and $x \simeq \gamma_1 - \alpha$ with $\alpha = \left(\frac{\gamma_2}{3\gamma_1}\right)^{1/3}$.

b) L_2 is at at $y = z = 0$, and $x \simeq \gamma_1 + \alpha$.

c) L_3 is at $y = z = 0$, and $x \simeq \gamma_1 - 2 + \frac{7}{4}\alpha^3$.

d) L_4 is at $x = \frac{1}{2} - \gamma_2$, $y = \frac{\sqrt{3}}{2}$, $z = 0$.

e) L_5 is at $x = \frac{1}{2} - \gamma_2$, $y = -\frac{\sqrt{3}}{2}$, $z = 0$.

- By slightly departing from $L_{j=1,2,3}$, show that those Lagrange points are unstable.

- Doing the same, show that $L_{4,5}$ are stable for $\gamma_2 < \gamma_{cr}$. Find numerically γ_{cr} (Hint: the value of γ_{cr} is rather small, less than 0.05; the difference between stable and unstable regimes will be very clear from the evolution.)

Additonal Notes

- If you use Mathematica, you will find especially useful the command NDSolve, which permits the solution of sets of differential equations. For example the following Mathematica code solves the system of equations $\ddot{x} = a\dot{x} - x$, $\ddot{y} = -a\dot{y} - by$ for some initial conditions, and plots the resulting (x, y) trajectories in a movie:

tmax=5;

a=0.1;b=0.5;x0=0.5;y0=0.3;

sol = NDSolve[{x"[t] == -a*x'[t] - x[t], y"[t] == -a*y'[t] - b*y[t],

x[0] == x0, y[0] == y0, x'[0] == 0.0, y'[0] == 0.0},

{x[t],y[t], x'[t], y'[t]}, {t, 0, tmax}];

nfr= 200; Lmax = 1; tmax = 25;

ListAnimate[Table[ParametricPlot[Evaluate[{x[t], y[t]} /. sol], {t, 0, n*tmax/nfr},

PlotRange -> {{-Lmax, Lmax}, {-Lmax, Lmax}}], {n, 1, nfr}],

AnimationRunning -> False]

- If you use C or FORTRAN, you will have first to transform the set of equations into a set of first-order differential equations, by simply introducing the velocities $(v_x, v_y, v_z) = \frac{d}{dt}(x, y, z)$. Calling $\vec{W} = (x, y, z, v_x, v_y, v_z)$, you may express the set of 6 coupled first-order differential equations in the form: $\frac{d}{dt}\vec{W} = \vec{F}[\vec{W}]$. If you know the solution at time t, you may evolve to a time $t + dt$ by employing the so-called 4th order Runge-Kutta method:

$$\vec{W}[t + dt] = \vec{W}[t] + \frac{dt}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \qquad (2)$$

with $\vec{k}_1 = \vec{F}[\vec{W}[t]]$, $\vec{k}_2 = \vec{F}[\vec{W}[t] + \vec{k}_1 dt/2]$, $\vec{k}_3 = \vec{F}[\vec{W}[t] + \vec{k}_2 dt/2]$, and $\vec{k}_4 = \vec{F}[\vec{W}[t] + \vec{k}_3 dt]$. You just need to start with the initial conditions $\vec{W}[0]$. Note that you should choose dt small enough to avoid numerical instability.