Analytische Mechanik und spezielle Relativitätstheorie

2. Computerübung **WS 14/15 Abgabetermin: 15.1.2014**

Vorlesung: Luis Santos – Übungen: Andreas Deser – Computerübungen: Xiaolong Deng

Hénon-Heiles System

The Hénon-Heiles Hamiltonian describes the motion of stars around the center of a smooth cylindrically symmetric galaxy, assuming the motion is restricted to the xy plane (we choose the mass $m = 1$):

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y),$$

with

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

In 1964 the astronomers Michel Hénon and Carl Heiles discovered that this potential, which also provides a simple model for a pair of nonlinearly coupled oscillators, yields regular orbits for some initial conditions and irregular, chaotic orbits for other initial conditions. This model is hence an excellent example to study the onset of chaos in conservative systems.

- Plot the Hénon-Heiles potential $V(x, y)$ in the vicinity of $(0, 0)$. You may employ in Mathematica the instruction ContourPlot or Plot3D. You will see that the potential has a minimum at $(0, 0)$, and that along the y direction there is an energy maximum at $(0, 1)$. You may easily see this by plotting $V(0, y)$ using the instruction Plot. What is the energy of this maximum? What is then the criterion the system energy E must fulfill such that the orbits cannot escape the vicinity of $(0, 0)$?

Using the Hamilton equations, one obtains two-coupled second-order differential equations for x and y:

$$\begin{aligned} \ddot{x} &= -x - 2xy, \\ \ddot{y} &= -y - x^2 + y^2. \end{aligned}$$

Note that the energy E is a conserved quantity. In order to solve the equations above one needs of course four initial conditions, namely $x(0)$, $y(0)$, $\dot{x}(0)$, and $\dot{y}(0)$. For a fixed energy E these initial conditions are not independent from each other, since $E = \frac{1}{2}(\dot{x}(0)^2 + \dot{y}(0)^2) + V(x(0), y(0))$.

- Write a code to solve the Hénon-Heiles equations. This may be done exactly in the same way as for the first computer exercise, i.e. employing NDSolve in Mathematica, or using Runge-Kutta if you use FORTRAN or C.

Since the phase space is 4-dimensional it is very convenient to use the idea of Poincaré section. One just plot the values of y and $\dot{y}$ when $x = 0$. These points build a two-dimensional plot, or Poincaré section.

- Plot the Poincaré section (for $x = 0$) for $E = 1/8$ and $x(0) = 0$, $y(0) = 0$, and $\dot{y}(0) = 0$ (see the additional notes at the end). What can you say about the motion? Is it regular (close elliptical curves) or chaotic (a mess of irregularly distributed points)? What about $x(0) = 0$, $y(0) = 0.2$, and $\dot{y}(0) = 0.2$. Is the motion now chaotic or regular?

For a fixed energy$E$, you may easily generate random initial conditions using the code in the additional notes at the end. If one combines the results for different initial conditions one may obtain a complete plot of the Poincaré section, which shows in general regions of regular and chaotic motion.

(Note: One should use different initial conditions, since especially for low E if one is in a regular region one cannot go out of it (you should have seen this in the previous point!) and hence one cannot explore other regions of the Poincare section. With different initial conditions this problem is avoided, and one can plot the whole Poincaré section.)

- Plot the Poincaré section for $E = 1/12$. You should see that there are four regions with elliptical orbits. In the center of these regions one has an elliptical fixed point. Can you determine approximately the fixed elliptical points?

- You should also find that in the frontier between the regular regions one has hyperbolic points (e.g. one of the hyperbolic points lies along the line $\dot{y} = 0$). Plot this hyperbolic region in detail, and convince yourself that this is indeed an hyperbolic point.

- Plot now the Poincaré section for $E = 1/8$. Can you determine the regions of chaotic and regular motion?

- Do the same with $E = 1/6$ (E cannot be larger than $1/6$, and you should know by now why). You should see the (almost complete) destruction of the regions of regular orbits (can you see still some regular regions?)

Additional Notes

(i) You may easily evaluate with Mathematica the Poincaré section. You just first use NDSolve to find the solution of the Hénon-Heiles equations from $t = 0$ to t_{max}. You may then use:

```
vec = {};
For[t0 = 0, t0 <= tmax, t0 += tmax/nmax,
  {
  x0 = (Evaluate[x[t] /. sol] /. t -> t0)[[1]];
  If[Abs[x0] < ϵ,
    {
    y0 = (Evaluate[y[t] /. sol] /. t -> t0)[[1]];
    y0p = (Evaluate[y'[t] /. sol] /. t -> t0)[[1]];
    vec = Append[vec, {y0, y0p}];
    }];
  }];
```

In this code t_{max} is the maximal time calculated in NDSolve; n_{max} is the number of time

steps (we calculate the Poincaré section only in those times); ϵ is a very small number, say 10^{-4}; the vector *vec* stores the $(y, \dot{y})$ points of the Poincaré section. *sol* stores the NDSolve solution, i.e.: *sol=NDSolve[...]* (recall the first computer exercise).

You may use the command *ListPlot[vec]* to plot the Poincaré section.

(ii) You may evaluate quite easily initial conditions, $x(0) = 0$, $\dot{x}(0)$, $y(0)$, and $\dot{y}(0)$, using the following Mathematica sub-code:

```
pyi = (RandomReal[]*2 - 1.)*Sqrt[2*En];
yi = (RandomReal[]*2 - 1.)*2*Sqrt[2*En];
check = 2*En - pyi^2 - yi^2 + 2*yi^3/3;
While[check < 0,
  {
  yi = (RandomReal[]*2 - 1.)*2*Sqrt[2*En];
  check = 2*En - pyi^2 - yi^2 + 2*yi^3/3;
  }];
pxi=Sqrt[check];
```

where *En* is the system energy, *pyi* is $\dot{y}(0)$, *yi* is $y(0)$, and *pxi* is $\dot{x}(0)$ (we demand $x(0) = 0$). Note that in this code we demand that $\dot{x}(0) = \sqrt{2E - \dot{y}(0)^2 - 2V(x(0), y(0))}$ is a real number.