Exercise 1: Bogoliubov transformation for two modes (8 P)

Consider two different types of bosonic particles, **a** and **b**, which interact by a contact interaction with each other but not between themselves. The Hamiltonian is then given by

$$H = \int \mathrm{d}^3r \left\{ \Psi^\dagger(\vec{r}) \left(\frac{-\hbar^2\nabla^2}{2m_a} \right) \Psi(\vec{r}) + \Phi^\dagger(\vec{r}) \left(-\frac{\hbar^2\nabla^2}{2m_b} \right) \Phi(\vec{r}) \right\} + U_0 \int \mathrm{d}^3r \Psi^\dagger(\vec{r})\Phi^\dagger(\vec{r})\Phi(\vec{r})\Psi(\vec{r}),$$

with $\Psi(\vec{r})$ and $\Phi(\vec{r})$ the field operators for the particles **a** and **b**, respectively.

(a) Write the Hamiltonian in momentum space. (1 P)

(b) Assuming that without interactions both components are condensed (in $\vec{k} = 0$) and are characterized by an overall density n_a and n_b, write (neglecting constant terms) the Hamiltonian to the first non-vanishing order in perturbation theory. (2 P)

(c) Diagonalize the Hamiltonian using a proper Bogoliubov transformation. (4P)

(Hint: Note that now there will be two Bogoliubov modes. If you did it right, you should get two different types of excitations

$$\epsilon_k^{\alpha,\beta} = \pm\frac{U_0\Delta n}{2} + \sqrt{\frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m} + 2U_0 n\right) + \frac{U_0^2\left(\Delta n\right)^2}{4}},$$

when $m_a = m_b = m$, $n_a = n + \Delta n/2$ and $n_b = n - \Delta n/2$.)

(d) What happens when $k \to 0$? Show that the spectrum for $\Delta n \neq 0$ is not linear for $k \to 0$ (i.e. the spectrum is not phonon-like), but quadratic with an effective mass $m^* = m\frac{\Delta n}{2n}$, i.e. the spectrum for both excitations at low k is of the form *constant* $+ \frac{\hbar^2k^2}{2m^*}$. (1 P)

Exercise 2: Klein-Gordon equation in Schrödinger form (8 P)

In the theory lecture we have seen that the Klein-Gordon equation $\left[\Box + (\frac{mc}{\hbar})^2\right]\Phi = 0$ is of 2nd order in time. We can however transform the Klein-Gordon equation into a first-order equation in time, by defining the vector $\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$, such that

$$\Phi = \varphi + \chi,$$
$$\frac{\partial\Phi}{\partial t} = \frac{-imc^2}{\hbar}(\varphi - \chi).$$

(a) Show that the system of coupled equations

$$i\hbar\frac{\partial}{\partial t}\varphi = \frac{-\hbar^2}{2m}\nabla^2(\varphi+\chi) + mc^2\varphi,$$
$$i\hbar\frac{\partial}{\partial t}\chi = \frac{+\hbar^2}{2m}\nabla^2(\varphi+\chi) - mc^2\chi,$$

is equivalent to the KG-equation. (1 P)

(b) Express the coupled equations in a compact form

$$i\hbar\frac{\partial}{\partial t}\Psi = H_f\Psi.$$

Express H_f using the Pauli matrices. (2 P)

(c) Express the density $\rho = \frac{i\hbar}{2mc^2}\left(\Phi^*\frac{\partial\Phi}{\partial t} - \Phi\frac{\partial\Phi^*}{\partial t}\right)$ as a function of φ and χ. (1 P)

(d) Let's consider free particles in the representation $\Psi = A\begin{pmatrix}\varphi_0\\ \chi_0\end{pmatrix}\mathrm{e}^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)}$ where A is a normalization constant. Find the dispersion $E(p)$, and the components. φ_0 and χ_0. (2 P)
(Hint: you should have solutions with $E > 0$ and solutions with $E < 0$.)

(e) What happens with the solutions in the non-relativistic limit? Have a look to the density ρ for the solutions with $E > 0$ and for those with $E < 0$. In one case you should get ρ *positive* and in the other *negative*. The density must be interpreted accordingly as charge density. (2 P)