
Exercise 1: Angular momentum (2 P)

Show that $\vec{J} = \vec{L}\mathbb{1} + \frac{\hbar}{2}\vec{\Sigma}$ commutes with the Hamilton-Dirac operator $H = -i\hbar c\vec{\alpha} \cdot \vec{\nabla} + \beta mc + V(r)$.

Exercise 2: Landau levels (8 P)

The time-independent Dirac equation describing a spin-1/2 particle of mass m and charge e in a static magnetic field with vector potential $\vec{A}$ is given by

$$E\Psi = \left\{ c\vec{\alpha} \cdot \left(-i\hbar\vec{\nabla} - e\vec{A} \right) + \beta mc^2 \right\} \Psi. \tag{1}$$

(a) Verify that

$$\left[\vec{\alpha} \cdot \left(-i\hbar\vec{\nabla} - e\vec{A} \right) \right]^2 = \left(-i\hbar\vec{\nabla} - e\vec{A} \right)^2 \mathbb{1} - e\vec{\Sigma} \cdot \vec{B}, \tag{2}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$. (2 P)

(b) For the particular case $\vec{B} = (0, Bx, 0)$ (Landau gauge), show, by considering solutions of the form

$$\Psi = \mathrm{e}^{i(p_y y + p_z z)/\hbar} u(x), \tag{3}$$

that the energy eigenvalues E of a relativistic electron in a constant magnetic field $\vec{B} = B\vec{e}_z$ are given by:

$$E_{n,\pm}^2 = p_z^2 c^2 + m^2 c^4 + e\hbar Bc^2(2n + 1 \pm 1). \tag{4}$$

These are the so-called Landau levels. (4 P)
[Hint: At some point you will find that part of the Hamiltonian looks like the Hamiltonian of an harmonic oscillator.]

(c) How is the non-relativistic limit? (2 P)

Exercise 3: Weyl equations (8 P)

Relativistivc quantum mechanics allows for the existence of massless particles. Here we will analyze massless spin-1/2 particles (applicable for the case of neutrinos). For this analysis it is useful to introduce a different representation of the Dirac matrices, the so-called Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}. \tag{5}$$

(a) Show that this representation fulfills indeed the anticommutation criterion for the Dirac matrices. (2 P)

(b) For the massless case the Dirac equation becomes

$$i\hbar\partial_t \Psi(\vec{r},t) = -i\hbar c\vec{\alpha}\cdot\vec{\nabla}\Psi(\vec{r},t). \tag{6}$$

Show that the Hamilton-Dirac operator $H = -i\hbar c\vec{\alpha}\cdot\vec{\nabla}$ commutes with $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. (1 P)

(c) From (b) we know now that if $\Psi(\vec{r},t)$ is a solution of the Dirac equation, also $\left(\frac{\mathbb{1}\pm\gamma^5}{2}\right)\Psi(\vec{r},t)$ is a solution. Using this show that we may find solutions $\Psi_L = \begin{pmatrix} 0 \\ \psi_L^w \end{pmatrix}$, $\Psi_R = \begin{pmatrix} \psi_R^w \\ 0 \end{pmatrix}$ such that

$$\begin{aligned} i\hbar\partial_t\psi_L^w(\vec{r},t) &= i\hbar c\vec{\sigma}\cdot\vec{\nabla}\psi_L^w(\vec{r},t), && (7)\\ i\hbar\partial_t\psi_R^w(\vec{r},t) &= -i\hbar c\vec{\sigma}\cdot\vec{\nabla}\psi_R^w(\vec{r},t) && (8) \end{aligned}$$

These are the so-called Weyl equations. Note that the $\psi_{L,R}^w$ are spinors of 2-components. (2 P)

(d) Show that under parity (spatial inversion) the Ψ_L and Ψ_R solutions get interchanged. (1 P)

(e) Using that for $m = 0$, $E = c|\vec{p}|$, and employing functions with well defined momentum

$$\begin{aligned} \psi_L^{w,\vec{p}(\vec{r},t)} &= \mathrm{e}^{-i\vec{p}\cdot\vec{r}/\hbar}\mathrm{e}^{-iEt/\hbar}\chi, && (9)\\ \psi_R^{w,\vec{p}(\vec{r},t)} &= \mathrm{e}^{i\vec{p}\cdot\vec{r}/\hbar}\mathrm{e}^{-iEt/\hbar}\chi, && (10) \end{aligned}$$

show that

$$\begin{aligned} S_p\Psi_L &= -\frac{\hbar}{2}\Psi_L, && (11)\\ S_p\Psi_R &= \frac{\hbar}{2}\Psi_R, && (12) \end{aligned}$$

where $S_p = \frac{\hbar}{2|\vec{p}|}\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix}$ is the helicity operator. The particle with helicity $-\hbar/2$ is the neutrino, and that with $+\hbar/2$ is the antineutrino. (2 P)