Exercise 1: Solution of the free Dirac equation in spherical coordinates (8P)

In this exercise you will solve the Dirac equation in free space in spherical coordinates. We are interested in the solutions $\Psi^l_{j,m_j}(\chi, \mathbf{r})$ for a given $\chi^2 \equiv \frac{E^2-m^2c^4}{\hbar^2c^2}$, with E the energy.

(a) Start with the radial equations of p. 96 of the script, but now without Coulomb potential. Let us call $A = \frac{iG}{r}$ and $B = \frac{F}{r}$. Eliminating B obtain the equation for A, which should have the form of a spherical Bessel equation. (3P)

(b) Show that the solution of A is of the form $A(r) = c_0 j_l(\chi r)$, where c_0 is a constant and $j_l(z)$ is a spherical Bessel function. (1P)

(c) Obtain B. (2P)
(Hint: Here you will have to use the properties of the spherical Bessel function $j_l(z)$. For the particular case of $l = j - 1/2$ you should get $B = \frac{-i\hbar c\chi}{E+mc^2} c_0 j_{l+1}(\chi r)$.)

(d) The results from (b) and (c) are up to a constant c_0 that you must determine by normalization, imposing

$$\int d^3r \left(\Psi^l_{j,m_j}\right)^* (\chi, \mathbf{r}) \cdot \Psi^l_{j',m'_j}(\chi', \mathbf{r}) = \delta(\chi, \chi')\delta_{j,j'}\delta_{m_j,m'_j}$$

. (Hint: If you do it right you should get $c_0^2 = \frac{2\chi^2}{\pi}\frac{E+mc^2}{2mc^2}$.) (2P)

Exercise 2: Dirac equation with a box potential (8P)

In this exercise, you will obtain the solution of the Dirac equation with a box potential

$$V(r) = \begin{cases} V_0 & r > R \\ 0 & r < R \end{cases} \tag{1}$$

We are interested in the case of bound states, i.e. solutions with energy E, such that $E < mc^2 + V_0$. Note that you have two regions: region I for $r < R$ and region II for $r > R$. The solution for region I is the same as that of exercise 1.

(a) You have to repeat now for region II, but now instead of E you have $E - V_0 < mc^2$. Show that in that case A fulfills a modified spherical Bessel equation. (3P)
(Hint: Instead of dependending on χ, the solutions depend now on $\bar{\chi} = \frac{1}{\hbar c}\sqrt{(mc^2 + V_0)^2 - E^2}$.

(b) Discuss, using the properties of the modified spherical Bessel functions, why the solution must be of the form $A = c_1 k_l(\bar{\chi} r)$, where c_1 is a constant, and $k_l(z)$ is a modified spherical Bessel function of the second kind. (2P)

(c) Consider the particular case of $l = 0$. Use the continuity of the solution, i.e. equate the logarithmic derivative $\frac{1}{A}\frac{dA}{dr}$ for both regions at $r = R$, to show that the energy fulfills the condition $\tan \chi R = -\frac{\chi}{\bar{\chi}}$. (3P)