Exercise 1: Symmetrization and anti-symmetrization operators

The symmetrization and anti-symmetrization operators are represented in terms of the permutation operator P_v by

$$S_\pm \equiv \frac{1}{\sqrt{N!}} \sum_v^{N!} (\pm 1)^{\epsilon_v} P_v, \tag{1}$$

where the summation is performed over all $N!$ permutations and ϵ_v is the parity to the permutation v.

(a) Show that

$$P_v S_\pm = (\pm 1)^{\epsilon_v} S_\pm. \tag{2}$$

(b) Show that

$$S_\pm^2 = \sqrt{N!} S_\pm. \tag{3}$$

(c) Show that

$$S_- S_+ = 0. \tag{4}$$

Exercise 2: Creation, annihilation and commutation rules for bosons

For bosons we have the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$.

(a) Show that

$$\left[\hat{a}, \left(\hat{a}^\dagger\right)^N\right] = N \left(\hat{a}^\dagger\right)^{N-1}. \tag{5}$$

(b) Show that

$$\hat{a}\hat{n}^m = (\hat{n}+1)^m \hat{a}, \qquad \text{with } \hat{n} = \hat{a}^\dagger \hat{a}. \tag{6}$$

(c) Let $\hat{H} = \sum_i \epsilon_i \hat{a}^\dagger \hat{a}$, where $\hat{a}_i$, $\hat{a}_i^\dagger$ are the annihilation, creation operators for mode i. Show that

$$\hat{a}_i(t) = \mathrm{e}^{i\hat{H}t/\hbar} \hat{a}_i \mathrm{e}^{-i\hat{H}t/\hbar} = \mathrm{e}^{-i\epsilon_i t/\hbar} \hat{a}_i \tag{7}$$

in two different ways:

(1) Use the results of (b) and commutation relations.

(2) Solve the Heisenberg equation $i\hbar \partial_t \hat{a}_i = [\hat{a}_i, \hat{H}]$.

Exercise 3: Coherent states

Let $\hat{D}(\alpha) = \mathrm{e}^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$; we define $\hat{D}(\alpha)|0\rangle \equiv |\alpha\rangle$. We want to show that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. For this we need:

(a) Show that $\hat{a}\mathrm{e}^{\alpha\hat{a}^\dagger} = \mathrm{e}^{\alpha\hat{a}^\dagger}(\hat{a}+\alpha)$.

(b) The Baker-Hausdorff formula says that if any two operators $\hat{A}$ and $\hat{B}$ are such that

$$\left[\left[\hat{A},\hat{B}\right],\hat{A}\right] = \left[\left[\hat{A},\hat{B}\right],\hat{B}\right] = 0, \tag{8}$$

then

$$\mathrm{e}^{\hat{A}+\hat{B}} = \mathrm{e}^{-\left[\hat{A},\hat{B}\right]/2}\mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}}. \tag{9}$$

Use this formula to show that

$$\hat{D}(\alpha) = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\hat{a}^\dagger}\mathrm{e}^{-\alpha^*\hat{a}}, \tag{10}$$

$$\hat{D}^\dagger(\alpha) = \mathrm{e}^{|\alpha|^2/2}\mathrm{e}^{\alpha^*\hat{a}}\mathrm{e}^{-\alpha\hat{a}^\dagger}. \tag{11}$$

(c) Using the results of (a) and (b) show that

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a}+\alpha. \tag{12}$$

(d) Using the results of (c) you can now show that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. These states $|\alpha\rangle$ are the so-called coherent states. They play a very important role in quantum mechanics.