Exercise 1: The Klein-Gordon equation in the non-relativistic limit

Show that the Klein-Gordon equation, $\left(\Box + (\frac{mc}{\hbar})^2\right)\Psi = 0$, becomes the Schrödinger equation in the non-relativistic limit. (Hint: use $\Psi = \varphi \mathrm{e}^{-imc^2t/\hbar}$, and consider $E' = E - mc^2 \ll mc^2$ where E is the total energy.)

Exercise 2: The minimal coupling/substitution

Show that the Hamiltonian function

$$H = \frac{\left(\vec{p} - q\vec{A}(\vec{x},t)\right)^2}{2m} + q\varphi(\vec{x},t) \tag{1}$$

provides the correct equation of motion $m\ddot{\vec{x}} = q(\vec{E} + \dot{\vec{x}} \times \vec{B})$ of a particle in the presence of an electromagnetic field. The substitution $H = \frac{p^2}{2m} \longrightarrow \frac{(\vec{p}-q\vec{A})^2}{2m} + q\varphi$ is the so-called minimal substitution or minimal coupling, which we have employed at various points in the theory lecture.

Exercise 3: Gauge transformation of Klein-Gordon and Dirac equations

Show that the minimal coupling leads to an invariance of both the Klein-Gordon equation and the Dirac equation against gauge transformations, i.e. if we make a gauge transformation $A'_\mu = A_\mu + \partial_\mu \chi$, if Ψ is the solution with A_μ, then $\Psi' = \mathrm{e}^{-iq\chi/\hbar}\Psi$ is the solution with A'_μ.

Exercise 4: Dirac matrices

(a) Show that the 4×4 matrices

$$\alpha^j \equiv \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \text{ where } j = 1,2,3, \tag{2}$$

and

$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \tag{3}$$

fulfill the anticommutation relations $\{\alpha^j, \alpha^k\} = 2\delta^{jk}\mathbb{I}$, $\{\alpha^j, \beta\} = 0$. As always $\sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

(b) Show that $\gamma^0 \equiv \beta$, $\gamma^j \equiv \beta\alpha^j$ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}$.