*Exercise 1: Angular eigen-spinors*

In the theory lecture we have introduced the eigenspinors of the angular act of the Dirac equation:

$$\rho^{(+)}_{j,m_j} = \begin{pmatrix} \sqrt{\frac{l+m_j+1/2}{2l+1}} Y_{l,m_j-1/2}(\theta,\phi) \\ \sqrt{\frac{l-m_j+1/2}{2l+1}} Y_{l,m_j+1/2}(\theta,\phi) \end{pmatrix}, \qquad l = j-1/2$$

$$\rho^{(-)}_{j,m_j} = \begin{pmatrix} \sqrt{\frac{l-m_j+1/2}{2l+1}} Y_{l,m_j+1/2}(\theta,\phi) \\ -\sqrt{\frac{l+m_j+1/2}{2l+1}} Y_{l,m_j-1/2}(\theta,\phi) \end{pmatrix}, \qquad l = j+1/2$$

In the lecture we employed the important property

$$(\vec{\sigma}\cdot\hat{x})\rho^{(-)}_{j,m_j} = \rho^{(+)}_{j,m_j}. \tag{1}$$

Show that this property is true. (Hint: to show this you will probably need to use the properties of the spherical harmonics and the Clebsch-Gordan coefficients.)

*Exercise 2: Zitterbewegung in the Heisenberg picture*

Consider the Dirac equation in free space

$$H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2.$$

We are interested here in having a look to the Zitterbewegung from the perspective of the Heisenberg picture, in which the dynamics of an operator $\hat{O}(t)$ is given by the corresponding Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}(t) = \frac{-i}{\hbar}\left[\hat{O}(t), \hat{H}\right].$$

It should be clear that the momentum $\vec{p}$ is conserved, and that $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = c\vec{\alpha}(t)$.

(a) Show that $\frac{\mathrm{d}\vec{\alpha}}{\mathrm{d}t} = \frac{-2i}{\hbar}\left[c\vec{p} - H\vec{\alpha}(t)\right]$. (Hint: use the anticommutation properties of $\vec{\alpha}$ and $\beta$.)

(b) Solve the equation for $\vec{\alpha}(t)$. If you do it right you should get

$$\vec{\alpha}(t) = c\frac{\vec{p}}{H} + \mathrm{e}^{2iHt/\hbar}\left[\vec{\alpha}(0) - c\frac{\vec{p}}{H}\right].$$

(Hint: as for usual ordinary differential equation, solve first the homogeneous equations, and then find a particular solution.)

(c) Solve for $\vec{x}(t)$. If you do it right you should get

$$\vec{x}(t) = \vec{x}(0) - c^2\frac{\vec{p}}{H}t - \frac{i\hbar c}{2H}\left(\mathrm{e}^{2iHt/\hbar} - 1\right)\left[\vec{\alpha}(0) - \frac{c\vec{p}}{H}\right]$$

You see there the appearance of the Zitterbewegung. We will see now that indeed the Zitterbwegung results from the interference of solutions with positive and negative energy.

(d) Show that

$$(\vec{\alpha}(0) - c\vec{p}/H)\, H + (\vec{\alpha}(0) - c\vec{p}/H)\, H = 0, \tag{2}$$

and hence

$$\mathrm{e}^{iHt/\hbar}\, (\vec{\alpha}(0) - c\vec{p}/H) = (\vec{\alpha}(0) - c\vec{p}/H)\, \mathrm{e}^{-iHt/\hbar} \tag{3}$$

(Hint: from the form of $H$, how does $H\vec{\alpha}$ relate to $\vec{\alpha}H$?)

(e) We are interested in the average position

$$\langle \vec{x}(t) \rangle \equiv \int \Psi^\dagger(0, \vec{x}) \vec{x}(t) \Psi(0, \vec{x}) \mathrm{d}^3 x, \tag{4}$$

where

$$\Psi(0, \vec{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{i\vec{p}\cdot\vec{x}}\, (\Phi_+(\vec{p}) + \Phi_-(\vec{p})) \tag{5}$$

is the wavepacket at $t = 0$, and $\Phi_\pm(\vec{p})$ are spinors with energy $\pm|E|$ and $|E(p)| = \sqrt{p^2c^2 + m^2c^4}$. From (c) one can see that

$$\langle \vec{x}(t) \rangle = \langle \vec{x}(0) \rangle + \langle \vec{x}_{ein}(t) \rangle + \langle \vec{x}_{ZB}(t) \rangle, \tag{6}$$

where $\langle \vec{x}_{ein}(t) \rangle \propto t$, and $\langle \vec{x}_{ZB}(t) \rangle$ is the contribution Zitterbewegung.

Show that the Zitterbewegung term $\langle \vec{x}_{ZB}(t) \rangle$ results from interference terms between $\Phi_+(\vec{p})$ and $\Phi_-(\vec{p})$, i.e. in the integral only crossed terms with $\Phi_+$ and $\Phi_-$ occur.

(Hint: you will have to use at some point the result of (d).)