
Exercise 10-1: Spin-orbit-coupling (6 points)

A spin-$\frac{1}{2}$ particle is in an orbit of angular momentum $l = 1$. The spin of the electron is coupled with the orbital angular momentum via $\hat{H} = \gamma \hat{\vec{L}} \cdot \hat{\vec{S}}$.

1. Write down the eigenenergies and eigenfunctions of $\hat{H}$ in the basis $|l, s; J, M\rangle \equiv |J, M\rangle$ of the operators $\{\hat{\vec{L}}^2, \hat{\vec{S}}^2, \hat{\vec{J}}^2, \hat{J}_z\}$, where $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$. Express the eigenfunction with the help of the Clebsch-Gordan coefficients in the eigenbasis $|l, s; m_l, m_s\rangle \equiv |m_l, m_s\rangle$ of the operators $\{\hat{\vec{L}}^2, \hat{\vec{S}}^2, \hat{L}_z, \hat{S}_z\}$. (3 points)

   Hint: The Clebsch-Gordan coefficients can be found at *http://en.wikipedia.org/wiki/Table_of_Clebsch-Gordan_coefficients#Formulation*, for instance. They are normaly given for $M \geq 0$ only. Here the following identity is useful: $\langle j_1, j_2; m_1, m_2 | j_1, j_2; J, M\rangle = (-1)^{J-j_1-j_2} \langle j_1, j_2; -m_1, -m_2 | j_1, j_2; J, -M\rangle$.

2. Let $|\psi(t = 0)\rangle = \sum_{m_l, m_s} \alpha_{m_l, m_s} |m_l, m_s\rangle$. Assume that the coupling coefficient γ is time dependent, $\gamma(t < 0) = 0$, $\gamma(t \in [0, t_0]) = 1$, and $\gamma(t > t_0 = 0)$.

   (a) Calculate the probability that the system is in a state $|m_l, m_s\rangle$ at $t > t_0$, formally expressed with the Clebsch-Gordan coefficients $\langle m_l, m_s | J, M\rangle$. (1.5 points)

   (b) If at $t = 0$ the coefficient $\alpha_{0,1/2}$ is nonvanishing only, what is the probability that the system is still in this state at $t > t_0$? Write down your answer as a function of γ and t. (1.5 points)

Exercise: Tensor-force (4 points)

The so-called tensor force between two spin-$\frac{1}{2}$ particles, which emerges from a dipole-dipole interaction, for instance, is given in position representation by an interaction energy of the form $\hat{V}(\vec{r}) = U(r)\hat{T}_{12}$, where

$$\hat{T}_{12} = \frac{(\hat{\vec{\sigma}}_1 \cdot \vec{r})(\hat{\vec{\sigma}}_2 \cdot \vec{r})}{r^2} - \frac{1}{3}(\hat{\vec{\sigma}}_1 \cdot \hat{\vec{\sigma}}_2).$$

Here $\hat{\vec{\sigma}}$ is the vector of Pauli-Matrices, and the indices $1, 2$ indicate the particle on which the matrices act. Let $\hat{\vec{J}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ with eigenstates $|\frac{1}{2}, \frac{1}{2}; J, M\rangle \equiv |J, M\rangle$, then $|0, 0\rangle$ is the singlet state and $|1, M\rangle$ with $M = 0, \pm 1$ are the triplet states..

Show that $\hat{T}_{12}$ can be written in the basis $\{|0, 0\rangle, |1, 0\rangle, |1, 1\rangle, |1, -1\rangle\}$ as

$$N \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2Y_{2,0} & -\sqrt{3}Y_{2,1} & -\sqrt{3}Y_{2,-1} \\ 0 & \sqrt{3}Y_{2,-1} & Y_{2,0} & \sqrt{6}Y_{2,-2} \\ 0 & \sqrt{3}Y_{2,1} & \sqrt{6}Y_{2,2} & Y_{2,0} \end{pmatrix},$$

with the spherical harmonics $Y_{l,m}$ and where $N = \frac{2}{3}\sqrt{\frac{4\pi}{5}}$.