
Exercise 1: Potential-well vs. delta-potential at low energies (6 points)

In this exercise you are supposed to consider a potential well of depth $-V_0$ in the range $[-a, a]$ at low positive energies E and investigate whether or not it is possible to obtain the same wave-functions for $|x| > a$ if the well is replaced by a delta-potential. We only consider the symmetric solutions here.

- Find the symmetric solutions of the potential-well for very small $E \gtrsim 0$. (2.5 points)

  Hint: You are supposed to find symmetric wave-functions with respect to the origin, not incoming and outgoing waves. You should obtain $\psi(c) \simeq C + Dx$ for $x > a$ (and similarly for $x < -a$) and calculate C/D as a function of a and $q = \sqrt{2mV_0/\hbar^2}$.

- Calculate the solutions of the potential $g\delta(x)$ for $E \gtrsim 0$. (1.5 points)

  The hint from the first part applies here as well.

- Show that the condition on g which has to be fulfilled if the eigenenergies in the two cases are required to be the same is given by (1.5 point)

$$g = -\frac{\hbar^2}{ma}\left(\frac{1}{1 + \frac{1}{qa\tan(qa)}}\right).$$

- Show that g behaves as $g \approx -\frac{\hbar^2}{2ma} qa \tan qa$ if the energy of a bound state of the well goes to 0 (if we change a or V_0). What happens then? (1.5 points)

  Hint: In the lecture the condition for the energy of bound states has been discussed.

Exercise 2: Tunneling with WKB (3 points)

Here you are supposed to calculate the absolute square of the transmission coefficient T for a potential, which takes the value $V(x) = \cos(\frac{\pi}{2b}x)^2$ in the interval $[-b, b]$ and 0 otherwise. You can use the WKB-formula from the lecture. We define $E = V(R)$, $R \leq b$.

- Calculate $|T|^2$ for $R \ll b$ and for $R \lesssim b$. (2 points)
- Compare the results. (1 point)

Hint: The substitution $y = \cos(\phi)$ could help in solving an integral.