Homeexercises Sheet 5 QUANTENMECHANIK I (SS08) Prof. Dr. Luis Santos
Hand in until 03.06.08 in the lecture. *Please state name and sheet number!*

Exercise 1: Generalized Kronig-Penny model (6 Points)

Consider the potential

$$V(x) = \frac{\hbar^2}{2m} \sum_{n=-\infty}^{\infty} \left[\frac{\lambda_1}{2a} \delta(x - 2na) + \frac{\lambda_2}{2a} \delta(x - (2n-1)a) \right],$$

where $a > 0$, λ_1 and λ_2 are constants. There are regions I_n with boundaries $2(n-1)a$ and $(2n-1)a$ and regions II_n with boundaries $(2n-1)a$ and $2na$. The wavefunction in the region I_n can in analogy to the lecture be written as $u_{I_n}(x) = A_n \cos k[x - (2n-1)a] + B_n \sin k[x - (2n-1)a]$, where $k^2 = 2mE/\hbar^2$.

1. Find the condition for the allowed positive energies $E \geq 0$ for general values of $\lambda_1 \neq \lambda_2$.

   (a) Write down $u_{II_n}(x)$ and show that

   $$\begin{aligned} A_n &= A_{n+1}[c + \frac{g_1}{2}s] - B_{n+1}[s + \frac{g_1}{2}(1-c)] \\ B_n &= A_{n+1}[s - \frac{g_1}{2}(1+c) - g_2 c - \frac{g_1 g_2}{2}s] + B_{n+1}[c + \frac{g_1}{2}s + g_2 s + \frac{g_1 g_2}{2}(1-c)], \end{aligned}$$

   where $s \equiv \sin 2ka$, $c \equiv \cos 2ka$, and $g_{1,2} = \frac{\lambda_{1,2}}{2ka}$. (2 points)

   (b) What is the other relation of A_n (B_n) with A_{n+1} (B_{n+1})? Use the answer to derive the condition

   $$\left| \frac{g_1 g_2}{4} + \left(1 - \frac{g_1 g_2}{4}\right) c + \left(\frac{g_1 + g_2}{2}\right) s \right| \leq 1$$

   for the energy. (2 points)

2. (a) Consider the case $g_1 = g > 0$, $g_2 = g/2$. Draw the regime of the allowed energies for the two cases $g = 1$ and $g = 10$. (1 point)

   (b) Show that for $g \to \infty$ only the energies $E_n \simeq \frac{\hbar^2 \pi^2}{2ma^2} n^2$ are allowed. (1 point)

Exercise 2: Deflected particle in the harmonic oszillator (4 points)

A particle of mass m is moving in the potential $V(x) = \frac{1}{2} m\omega^2 x^2$ of a harmonic oszillator with frequency ω. At time $t = 0$ the particle is in the state

$$\psi(x, 0) = \frac{1}{\sqrt{\sqrt{\pi} l}} e^{-(x-x_0)^2/2l^2} \quad \text{with} \quad l = \sqrt{\frac{\hbar}{m\omega}}.$$

1. Express $\psi(x, 0)$ as a linear combination of eigenfunctions of the harmonic oszillator.
   Hint: The formula $\int_{-\infty}^{\infty} dx \, e^{-(x-x_0)^2} H_n(x) = \sqrt{\pi} 2^n x_0^n$ should be helpful. (2 points)

2. Write down $\psi(x, t)$. How does the density $|\psi(x, t)|^2$ behave?
   Hint: The relation $e^{-q^2+2qy} = \sum_{n=0}^{\infty} \frac{q^n}{n!} H_n(y)$ should be of use. (2 points)