Exercise 1: Deflected particle in the harmonic oscillator (3 Points)

A particle of mass m is moving in an one-dimensional harmonic oscillator with frequency ω. At $t = 0$ the state of the system is given by

$$|\Psi(t=0)\rangle = e^{-i\hat{p}b/\hbar}|0\rangle,$$

where b is a constant and $|0\rangle$ is the ground-state of the harmonic oscillator.

1. Using the Heisenberg picture, calculate the expected value $\langle \hat{x} \rangle(t)$ for $t \geq 0$.
2. Consider $|\Phi\rangle = e^{-i\hat{p}b/\hbar}|\Psi\rangle$, and show that $\Phi(x+b) = \Psi(x)$ holds.

   Note: This is why the operator $e^{-i\hat{p}b/\hbar}|\Psi\rangle = \hat{T}_b$ is the so-called translation operator (by a length b).
   Hint: The Baker-Hausdorff lemma will be helpful.

Exercise 2: Correlation function (2 points)

Consider a function known as the correlation function, which is defined by

$$c(t) = \langle \hat{x}(t)\hat{x}(0) \rangle,$$

where $\hat{x}(t)$ is the position operator in the Heisenberg picture. Evaluate $c(t)$ explicitly for the ground-state of a one-dimensional harmonic oscillator.

Exercise 3: Schwinger-Boson representation (3 points)

Consider two independent harmonic oscillators, whit Hamiltonian operators $\hat{H}_A$ and $\hat{H}_B$, and creation and annihilation operators $(\hat{a}, \hat{a}^\dagger)$ and $(\hat{b}, \hat{b}^\dagger)$ respectively. The ladder operators satisfy the following commutation relations: $[\hat{a}, \hat{b}] = [\hat{a}^\dagger, \hat{b}^\dagger] = [\hat{a}, \hat{b}^\dagger] = [\hat{a}^\dagger, \hat{b}] = 0$. Let's to consider the operators $\hat{J}_+, \hat{J}_-, \hat{J}_z$ defined as:

$$\hat{J}_+ = \hbar\hat{b}^\dagger\hat{a}, \quad \hat{J}_- = \hbar\hat{a}^\dagger\hat{b} \qquad \text{where} \quad \hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \quad \hat{J}_z = \frac{\hbar}{2}(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a})$$

(this is the so-called Schwinger-Boson representation of the angular momentum operators).

1. Show that the operators $\hat{J}_\pm, \hat{J}_z$ fulfill the same commutation relations of the angular momentum operators, to do so, check the commutators $[\hat{J}_z, \hat{J}_\pm], [\hat{J}^2, \hat{J}_z], [\hat{J}_+, \hat{J}_-]$.
2. Consider the states $|n_A, n_B\rangle$ such that

   $$\hat{H}_A|n_A, n_B\rangle = \hbar\omega_A\left(n_A + \frac{1}{2}\right)|n_A, n_B\rangle \quad \text{and} \quad \hat{H}_B|n_A, n_B\rangle = \hbar\omega_B\left(n_B + \frac{1}{2}\right)|n_A, n_B\rangle.$$

   Show that $|n_A, n_B\rangle$ are eigenstates of $\hat{J}_z$ and $\hat{J}^2$, where $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$. Calculate the corresponding eigenvalues, what plays now the role of the quantum numbers l and m?

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Exercise 4: Heisenberg equation of motion (2 points)

Consider the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V_0 \cos^2 q\hat{x},$$

where m, V_0 and q are constants. Obtain the Heisenberg equation for the operators $\hat{p}$ and $\hat{x}$ and the equation of motion $\frac{d^2\hat{x}}{dt^2}$ =? (without solving it).