Homeexercises Sheet 8 QUANTENMECHANIK I (SS08) Prof. Dr. Luis Santos
Hand in until 24.06.08 in the lecture. *Please state name and sheet number!*

Exercise 1: Central potential (10 points)

The whole sheet is concerned with the following central potential:

$$V(r) = \frac{c}{r^2} + \frac{m\omega^2}{2} r^2$$

with the constants $c > 0$, m, and ω.

1. Sketch the potential in dependence of r, write down the corresponding radial equation for $R_{nl}(r)$ [where $\psi_{nlm}(\vec{r}) = R_{nl}(r) Y_{lm}(\theta, \phi)$] and transform it into a dimensionless equation with the help of the definitions $r = \rho\sqrt{\frac{\hbar}{m\omega}}$, $\tilde{c} = \frac{m}{\hbar^2} c$, and $\epsilon = \frac{2E}{\hbar\omega}$, where E is the energy which turns up in the radial equation. (0.5 points)

2. Transform the dimensionless equation into an equation for $u(\rho) = \rho R(\rho)$. (1 point)

   Hint: You should obtain an equation of the form $\frac{d^2u}{d\rho^2}(\rho) + F(\rho)u(\rho) = 0$, where $F(\rho)$ is a function of ρ.

3. Show that $u(\rho)$ behaves (i) in the limit $\rho \to \infty$ as $e^{-\rho^2/2}$ and (ii) in the limit $\rho \to 0$ as ρ^χ, where $\chi(\chi - 1) = 2\tilde{c} + l(l + 1)$. Argue that $\chi > 0$ has to hold. (1 point)

4. Use the ansatz $u(\rho) = \rho^\chi e^{-\rho^2/2} g(\rho)$ and show that $g(\rho)$ has to fulfil the equation

$$g'' + \frac{2}{\rho}(\chi - \rho^2)g' + (\epsilon - 2\chi - 1)g = 0 \quad \text{where} \quad g' = \frac{dg(\rho)}{d\rho}.$$

   (1 point)

5. Expand $g(\rho)$ as a power series, $g(\rho) = \sum_{n=0}^{\infty} a_n \rho^n$, and show that the coefficients have to fulfil the recursion relation (2 points)

$$\frac{a_{n+2}}{a_n} = \frac{2n - (\epsilon - 2\chi - 1)}{(n+2)(n+1) + 2\chi(n+2)}.$$

6. Show that the series diverges if the numerator does not vanish for some value of n, so that the series truncates. With this you should obtain the possible energy eigenvalues $E_n = \hbar\omega(n + \frac{1}{2} + \chi)$. (1 point)

   Hint: it might be helpful to consider the limit $n \to \infty$.

7. Express χ as a function of $\tilde{c}$ and l and show that for $c \to 0$ the energy values are given by $E_n = \hbar\omega(n + l + \frac{3}{2})$ (corresponding to the energy eigenvalues of the 3D harmonic oscillator). How many states have the same energy, so what is the degeneracy of an energy level? (1.5 points)

8. Calculate for $c > 0$ the ground state wavefunction $R_0(\rho)$ (without normalizing it) with the results of part 4 and 5. Calculate the maximum of the radial probability density $\rho^2|R_0(\rho)|^2$ and check if the radius of the maximal density agrees in general with the radius of the minimum of the potential. (2 points)