Exercise 9-1: Pauli-matrices (2.5 points)

Consider the operator

$$\hat{R}_{\vec{a}}(\theta) = \exp(-\frac{i}{2}\,\theta\,\hat{\vec{\sigma}}\cdot\hat{\vec{a}}) \quad \text{with} \quad \hat{\vec{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z),$$

where $\hat{\sigma}_{x,y,z}$ are the Pauli-matrices. In this exercise you are supposed to investigate this operator for a vector $\vec{a}$ in the $x-y$ plane, $\vec{a} = (-\sin\phi, \cos\phi, 0)$.

1. Write down the operator $\hat{R}_{\vec{a}}(\theta)$ in the matrix representation of the $\hat{S}_z$-basis. (1 point)

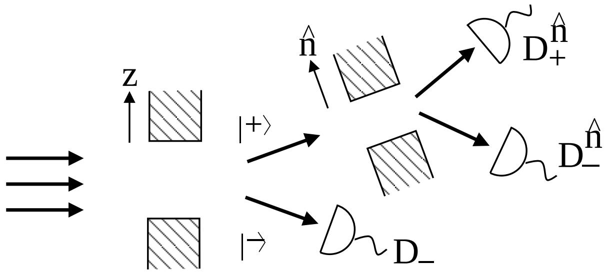
   Useful questions: how is the function of an operator defined? What is the Taylor-expansion of the sine and the cosine?

2. Let $\hat{\sigma}_z|+\rangle = |+\rangle$. Calculate $|\psi'\rangle = \hat{R}_{\vec{a}}(\theta)|+\rangle$. (0.5 points)

3. Compare $\langle+|\hat{\vec{\sigma}}|+\rangle$ with $\langle\psi'|\hat{\vec{\sigma}}|\psi'\rangle$. What is the effect of the operator $\hat{R}_{\vec{a}}(\theta)$? Support your answer with a sketch. (1 point)

Exercise 9-2: Double Stern-Gerlach experiment (2 Punkte)

A spin-$\frac{1}{2}$ particle is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle + |-\rangle\big)$, where $|\pm\rangle$ are the eigenstates of the z-component of the spin operator $\hat{S}_z$. The particle is first flying through a Stern-Gerlach apparatus in the z-direction, and the part which is deflected in the positive z-direction is then flying through a second Stern-Gerlach apparatus in the direction $\hat{n}$, which is parametrized by the polar angles θ and ϕ.



Obtain the probabilities for detecting the particle in the detectors D_-, $D_+^{\hat{n}}$, and $D_-^{\hat{n}}$.

Exercise 9-3: Magnetic spinning top I (3 points)

The Hamilton operator of a magnetic spinning top in a magnetic field $\vec{B}$ is given by

$$\hat{H} = \frac{1}{2\theta_1}(\hat{L}_x^2 + \hat{L}_y^2) + \frac{1}{2\theta_2}\hat{L}_z^2 - g\mu\,\vec{B}\cdot\hat{\vec{L}}.$$

We assume that $\theta_{1,2} > 0$ holds as well as $\vec{B} = B\hat{z}$ with $B > 0$ and $g\mu > 0$.

Turn over!

1. Obtain the eigenstates and eigenenergies of $\hat{H}$. (0.5 points)
2. Find the groundstate depending on $b \equiv \frac{2g\mu B\theta_1}{\hbar}$ and $R \equiv \frac{\theta_1}{\theta_2}$. (1.5 points)
3. For which value of B has the ground state the angular momentum $l = 1$? (0.5 points)
4. Under which condition is the groundstate energy degenerate? (0.5 points)

Exercise 9-4: Magnetic spinning top II (2.5 points)

This exercise concerns again the Hamilton operator of the magnetic spinning top. Here the magnetic field is given by $\vec{B} = B(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

1. Write down the Heisenberg-equations for the operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$. (1.5 points)
2. Find the solution of the equations $\langle\hat{L}_x\rangle(t)$ and $\langle\hat{L}_y\rangle(t)$ for $\theta = 0$ and $\theta_1 = \theta_2$ depending on the initial values $\langle\hat{L}_x\rangle(0)$ and $\langle\hat{L}_y\rangle(0)$. (1 point)