Sheet 5 (to be returned on 25.05.07 before 12:00)

Exercise 1: Modified Kronig-Penney potential (5 Points)

Consider the potential $V(x) = g\sum_{n=-\infty}^{\infty}\delta(x-na) - g\sum_{n=-\infty}^{\infty}\delta(x-(n-1/2)a)$, where $g > 0$, and $a > 0$ are constants. Consider positive energies $E > 0$, and obtain the condition for the allowed energy bands. (Hint: Pay a close attention to what is the periodicity of the potential. If one calls R_n the region $(n-1)a < x < na$, the wavefunctions $\psi_n(x)$ in R_n and $\psi_{n+1}(x)$ in R_{n+1} are related by $\psi_{n+1}(x) = \psi_n(x+a)e^{i\phi}$, where ϕ is a phase).

Exercise 2: Harmonic oscillator wavefunctions (3 Points)

Consider a particle of mass m described by a Gaussian wavefunction of the form:

$$\psi(x) = \frac{1}{\sqrt{\sqrt{\pi}l_0}}e^{-\frac{1}{2}\left(\frac{x}{l_0}\right)^2}$$

Assume an harmonic oscillator centered in $x = 0$ with a frequency ω, such that the oscillator length is $l = \sqrt{\hbar/m\omega} > l_0$. What is the probability to find the particle with an energy $E_n = \hbar\omega(n+1/2)$? (Hint: You will need the following integral $\int_{-\infty}^{\infty} dy e^{-y^2/2u} H_n(y) = (2\pi u)^{1/2}(1-2u)^{n/2}H_n(0)$, where $H_n(y)$ are the Hermite polynomials, and $H_{2n'}(0) = (-1)^{n'}2^{n'}(2n'-1)!!$ and $H_{2n'+1}(0) = 0$, where $(2n'-1)!! = 1\cdot 3\cdot 5\cdot\ldots(2n'-1)$.)

Exercise 3: Width of the harmonic oscillator wavefunctions (2 Points)

Calculate the variance $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ for the n-th level of a 1D harmonic oscillator centered in $x = 0$. (Hint: You will need the following integral $\int_{-\infty}^{\infty} dy e^{-y^2} H_n(y)^2 y^2 = 2^n n!\sqrt{\pi}(n+1/2)$).