Exercise 1: Periodic potential (2 Points)

Consider the following potential: $V(x) = g\sum_{n=-\infty}^{\infty}\delta(x-(na+c)) + U(x)$, where $U(x) = 0$, except for the regions $na - b < x < na$, where it is $U(x) = V_0$. Consider a particle with energy $E < V_0$. Express the wavefunctions in the different regions of the potential, and write down (without solving them) the equations that these wavefunctions must fulfill.

Exercise 2: Creation and destruction operators (3 Points)

Consider the n-th state ($|n\rangle$) of the harmonic oscillator

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$$

where $|0\rangle$ is the ground-state of the harmonic oscillator. We employ a slightly different definition for the creation and annihilation operators as that of the theory class, which is indeed the more standard way of defining these operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{2m\hbar\omega}},$$
$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{\hat{p}}{\sqrt{2m\hbar\omega}},$$

Note that conpared with the theory class $\hat{a} = \hat{A}/\sqrt{\hbar}$, and $\hat{a}^\dagger = \hat{A}^\dagger/\sqrt{\hbar}$, and hence $[\hat{a}, \hat{a}^\dagger] = 1$, and the harmonic oscillator Hamiltonian is of the form $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$.

- Show that $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.
- Show that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$. In order to show this, you will need the commutation relation of $\hat{a}$ and $\hat{a}^\dagger$.
- Calculate $\langle n|\hat{x}|m\rangle$ showing that it is zero except when $n = m \pm 1$.

Exercise 3: Heisenberg picture (3 Points)

Consider the following Hamiltonian Operator:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m}{2}\left(\omega_1^2\hat{x}^2 + \omega_2\hat{x} + \epsilon\right).$$

- Using the commutation relation of the position operator $\hat{x}$ and the momentum operator $\hat{p}$, obtain the Heisenberg equations for $\frac{d}{dt}\hat{x}$, and $\frac{d}{dt}\hat{p}$.
- Solve the Heisenberg equations to obtain the time-dependence of the operator $\hat{x}(t)$.

- Express the Hamiltonian operator as a function of the creation and annihilation operators (using the definiton of exercise 2). Write down (without solving it) the Heisenberg equation for $\frac{d}{dt}\hat{a}$.

Exercise 4: Heisenberg picture II (2 Points)

- Using the definition of commutator calculate the commutator $\left[\hat{p}, \frac{1}{x^2}\right]$.
- Consider the Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2 x^2 + \frac{A}{x^2},$$

  where A is a constant. Write down (without solving them) the Heisenberg equations for $\frac{d}{dt}\hat{x}$, and $\frac{d}{dt}\hat{p}$.