
Exercise 1: The Baker-Hausdorff formula

Consider two operators $\hat{A}$ and $\hat{B}$ that satisfy $[\hat{A},[\hat{A},\hat{B}]]=[\hat{B},[\hat{A},\hat{B}]]=0$.

- Show first that $[\hat{B},\hat{A}^n]=-n\hat{A}^{n-1}[\hat{A},\hat{B}]$.
- Consider $\hat{f}(x)=e^{\hat{A}x}e^{\hat{B}x}$. Show that

$$\frac{d\hat{f}(x)}{dx}=\left(\hat{A}+e^{\hat{A}x}\hat{B}e^{-\hat{A}x}\right)\hat{f}(x).$$

- Using the result of the first point, show that the expression obtained in the second point can be expressed as

$$\frac{d\hat{f}(x)}{dx}=\left(\hat{A}+\hat{B}+[\hat{A},\hat{B}]x\right)\hat{f}(x).$$

- Setting $\hat{f}(x)=e^{\hat{O}(x)}$, find the operator $\hat{O}(x)$, using the fact that $\hat{f}(0)=1$.
- Once you have found $\hat{O}(x)$, set $x=1$, and check that you obtain the Baker-Hausdorff formula that we have employed in the theory class.

Exercise 2: Squeezed states

Consider the squeezed state $|\alpha,\epsilon\rangle=\hat{D}(\alpha)\hat{S}(\epsilon)|0\rangle$. Let $\epsilon=re^{i\phi}$, where r is the squeezing factor. Using the properties of the displacement operator $\hat{D}(\alpha)$ and the squeezing operator $\hat{D}(\epsilon)$ that we have seen in the class, show that:

- $\langle\alpha,\epsilon|\hat{a}|\alpha,\epsilon\rangle=\alpha$
- $\langle\alpha,\epsilon|\hat{a}^\dagger|\alpha,\epsilon\rangle=\alpha^*$
- $\langle\alpha,\epsilon|\hat{a}^2|\alpha,\epsilon\rangle=\alpha^2-e^{2i\phi}\sinh r\cosh r$
- $\langle\alpha,\epsilon|\hat{a}^{\dagger 2}|\alpha,\epsilon\rangle=\alpha^{*2}-e^{-2i\phi}\sinh r\cosh r$

Exercise 3: Number average and number variance in squeezed states

Consider the squeezed state $|\alpha,\epsilon\rangle$. Let $\hat{n}=\hat{a}^\dagger\hat{a}$ the number of operator.

- Show that the average number of photons is $\langle\alpha,\epsilon|\hat{n}|\alpha,\epsilon\rangle=|\alpha|^2+\sinh^2 r$
- Show that the squared variance $(\Delta n)^2=\langle\hat{n}^2\rangle-\langle\hat{n}\rangle^2$ is $(\Delta n)^2=|\alpha\cosh r-\alpha^* e^{-2i\phi}\sinh r|^2+2\cosh^2 r\sinh^2 r$.
- Consider the case $|\alpha|^2$ very large. Show that $\Delta n\simeq\sqrt{\langle n\rangle}e^{-r}$, and hence that if $r>0$ one has a sub-Poissonian distribution, and for $r<0$ a super-Poissonian distribution.