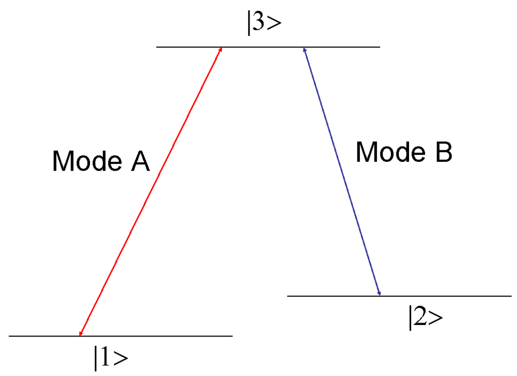
Exercise 1: The Λ system

Let us consider a 3 level atom with an electronic configuration as in the figure. This sort of system is called (for obvious reasons) a Λ system.



The level $|j\rangle$ has an energy E_j, with $j = 1, 2, 3$. The levels $|1\rangle$ and $|3\rangle$ are connected by a laser with frequency ω_A, whereas the levels $|2\rangle$ and $|3\rangle$ are connected by a laser with frequency ω_B. The coupling constant for the coupling $|j\rangle \leftrightarrow |3\rangle$ is g_{j3} with $j = 1, 2$ (assume both coupling constants as real).

- Write the Hamiltonian of the system in rotating wave approximation. Note that now the system is formed by the three electronic levels and the two electromagnetic modes.

- Show that the states $\{|n_A, n_B, 3\rangle, |n_A+1, n_B, 1\rangle, |n_A, n_B+1, 2\rangle\}$ form a closed family of states, i.e. that when we apply the Hamiltonian to any one of them we get a combination of other members of the family. The notation $|n_A, n_B, j\rangle$ means that the state has n_A photons in the mode A, n_B photons in the mode B, and the electron is in the level $|j\rangle$.

- Let $|\psi(t)\rangle = \psi_3(t)|n_A, n_B, 3\rangle + \psi_1(t)|n_A + 1, n_B, 1\rangle + \psi_2(t)|n_A, n_B + 1, 2\rangle$. Write the equations for $\dot{\psi}_j(t)$, with $j = 1, 2, 3$. Use the definition of the detunings $\Delta_{13} = \omega_A - (E_3 - E_1)/\hbar$, and $\Delta_{23} = \omega_B - (E_3 - E_2)/\hbar$.

Exercise 2: The dark state

Consider the system of the previous exercise, with $\Delta_{13} = \Delta_{23} = \Delta$. Let $\Omega_1 = g_{13}\sqrt{n_A + 1}$, $\Omega_2 = g_{23}\sqrt{n_B + 1}$.

- Show that the state

$$|D\rangle = \frac{\Omega_2|n_A + 1, n_B, 1\rangle - \Omega_1|n_A, n_B + 1, 2\rangle}{\sqrt{\Omega_1^2 + \Omega_2^2}}$$

  decouples from the rest of the states, i.e. there is no transfer between $|D\rangle$ and the rest of the states. This state is an example of a so-called *dark state*.

- Calculate the eigenstates and eigenenergies of the Hamiltonian of the Λ system. Note that the dark state is itself an eigenstate.

Exercise 3: Time evolution of the Λ system

Consider the Λ system with the assumptions of exercise 2.

- Obtain the state of the system $|\psi(t)\rangle$ for a given initial state $|\psi(0)\rangle$.
- Let $\psi(0) = |n_A + 1, n_B, 1\rangle$. What is the probability to find the atom after some time t in the electronic state $|j\rangle$, with $j = 1, 2, 3$?