Exercise 1: Symmetrically-ordered characteristic function for a squeezed state

Let us consider a squeezed state $|\alpha, r\rangle$, where α is a complex number, but r is real. The goal is to find the corresponding form of the symmetrically ordered characteristic function

$$\chi(\eta) = Tr\left\{\hat{\rho} e^{\eta\hat{a}^\dagger - \eta^*\hat{a}}\right\} = \langle\alpha, r|e^{\eta\hat{a}^\dagger - \eta^*\hat{a}}|\alpha, r\rangle.$$

Remember that $|\alpha, r\rangle = \hat{D}(\alpha)\hat{S}(r)|0\rangle$. In order to find $\chi(\eta)$ you will have to follow the following steps:

- First use the Baker-Hausdorff formula to show that

$$e^{\eta\hat{a}^\dagger - \eta^*\hat{a}} = e^{\eta\hat{a}^\dagger} e^{-\eta^*\hat{a}} e^{-|\eta|^2/2}$$

- Show that

$$\hat{D}(\alpha)^\dagger e^{\eta\hat{a}^\dagger} e^{\eta^*\hat{a}} \hat{D}(\alpha) = e^{\eta(\hat{a}^\dagger + \alpha^*)} e^{-\eta^*(\hat{a}+\alpha)}$$

- Show then that

$$\hat{S}(r)^\dagger e^{\eta\hat{a}^\dagger} e^{-\eta^*\hat{a}} \hat{S}(r) = e^{\eta(\cosh r\hat{a}^\dagger - \sinh r\hat{a})} e^{-\eta^*(\cosh r\hat{a} - \sinh r\hat{a}^\dagger)}$$

- Using again the Baker-Hausdorff formula, show that

$$e^{\eta(\cosh r\hat{a}^\dagger - \sinh r\hat{a})} = e^{\eta\cosh r\hat{a}^\dagger} e^{-\eta\sinh r\hat{a}} e^{-\frac{1}{2}\sinh r\cosh r\eta^2},$$

$$e^{-\eta^*(\cosh r\hat{a}^\dagger - \sinh r\hat{a})} = e^{\eta^*\sinh r\hat{a}} e^{-\eta^*\cosh r\hat{a}} e^{-\frac{1}{2}\sinh r\cosh r\eta^{*2}},$$

- Note that $e^{-\eta^*\cosh r\hat{a}}|0\rangle = |0\rangle$, and $\langle 0|e^{\eta\sinh r\hat{a}^\dagger} = \langle 0|$.
- Use again the Baker Hausdorff formula to show that:

$$e^{-\eta\sinh r\hat{a}} e^{\eta^*\sinh r\hat{a}^\dagger} = e^{\eta^*\sinh r\hat{a}^\dagger} e^{-\eta\sinh r\hat{a}} e^{-|\eta|^2\sinh^2 r}$$

- Now you should be able to calculate easily

$$\langle 0|e^{-\eta\sinh r\hat{a}} e^{\eta^*\sinh r\hat{a}^\dagger}|0\rangle$$

If you did all the calculation correctly you should retrieve:

$$\chi(\eta) = e^{\eta\alpha^* - \eta^*\alpha - |\eta|^2/2 - |\eta|^2\sinh^2 r - \frac{1}{2}\sinh r\cosh r(\eta^2 + \eta^{*2})}$$

Exercise 2: Wigner function of a squeezed state

We consider the state $|\alpha, r\rangle$ discussed in the first exercise. The goal now is to calculate the Wigner function associated to this state. Remember that

$$W(\beta) = \frac{1}{\pi^2}\int d^2\eta e^{\eta^*\beta - \eta\beta^*}\chi(\eta)$$

By writting everything in terms of real and imaginary parts, you will retrieve to Gaussian integrals (for η_r and for η_i). Remember that the Fourier transform of a Gaussian is other Gaussian:

$$\int_{-\infty}^{\infty} dx e^{-x^2/\sigma^2} e^{ikx} = \sqrt{\pi}\sigma e^{-k^2\sigma^2/4}$$

If we define $\beta = (x_1 + ix_2)/2$, and $\alpha = (X_1 + iX_2)/2$, show that

$$W(\beta) = \frac{2}{\pi} e^{-\frac{1}{2}\frac{(x_1 - X_1)^2}{e^{2r}}} e^{-\frac{1}{2}\frac{(x_2 - X_2)^2}{e^{-2r}}}$$

Exercise 3: P-Representation of a squeezed state

Try to evaluate the P-representation, $P(\beta)$, of a squeezed state. Remember that $\chi_N(\eta) = \chi(\eta) e^{|\eta|^2/2}$, and that $P(\beta) = \int d^2\eta e^{\eta^*\beta - \eta\beta^*} \chi_N(\eta)$. Where is the problem?