
Exercise 1: The optical Bloch equations

We consider an atom excited by a laser that may decay into the many modes of the (vacuum) electromagnetic field (i.e. the problem of resonance fluorescence).

- From the master equation find the optical Bloch equations for $\langle\hat{\sigma}_-\rangle$, $\langle\hat{\sigma}_+\rangle$, and $\langle\hat{\sigma}_z\rangle$.
- Show that the solutions of the master equation are those indicated at the end of page 101 in the script.
- Given the Bloch vector $\vec{v} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$, show that

$$\frac{d}{dt}|\vec{v}|^2 = -\gamma\left[|\vec{v}|^2 - 1 + (\langle\hat{\sigma}_z\rangle + 1)^2\right]$$

Exercise 2: Q-representation for the damped harmonic oscillator

Show that the Q-representation for a Fock state $|l\rangle$ is

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}e^{-|\alpha|^2}\frac{|\alpha|^{2l}}{l!}$$

Let us suppose that we have an harmonic oscillator in an initial Fock state with one excitation $|l = 1\rangle$, i.e. $\hat{\rho}(0) = |1\rangle\langle 1|$. The oscillator is assumed to be in contact with a reservoir with $\bar{n} = 0$, and damping constant γ (we use the same notation as in the theory lecture). Use the Green's function for the Fokker-Planck equation in the Q-representation for a damped harmonic oscillator $Q(\alpha,\alpha^*,t|\lambda,\lambda^*,0)$ (see script, p. 128) to find the Q-representation $Q(\alpha,\alpha^*,t)$ at any time $t > 0$.

To do the calculations you will need the following information:

- Remember that $Q(\alpha,\alpha^*,t) = \int d^2\lambda Q(\alpha,\alpha^*,t|\lambda,\lambda^*,0)Q(\lambda,\lambda^*,0)$.
- Employ the change of variables $r = |\lambda|$ and $\phi = arg(\lambda) - arg(\alpha) + \omega_0 t$.
- You will need that

$$\int_0^{2\pi} d\phi \exp\left[\left(\frac{2|\alpha|e^{-\gamma t/2}}{1-e^{-\gamma t}}\right) r\cos\phi\right] = 2\pi\sum_0^\infty \frac{1}{(k!)^2}\left[\frac{r|\alpha|e^{-\gamma t/2}}{1-e^{-\gamma t}}\right]$$

- You will also need the Gaussian integral

$$2\int_0^\infty r^{2k+3}e^{-r^2/(1-e^{-\gamma t})} = (k+1)!\left(1-e^{-\gamma t}\right)^{k+2}$$

- Finally, it will be also useful the property

$$\sum_{k=0}^{\infty} \frac{(k+1)!}{(k!)^2} x^k = e^x(1+x)$$

If you do the calculation correctly you should obtain:

$$Q(\alpha, \alpha^*, t) = \frac{1}{\pi} e^{-|\alpha|^2} \left[1 - e^{-\gamma t} + |\alpha|^2 e^{-\gamma t}\right]$$

Obtain the stationary state solution. Which kind of state is the stationary state? Remember the discussion about the Q-representation in p. 66 of the script.

The Quantum Optics team wishes you Merry Christmas!!