
Exercise 1: Operator ordering (11 points)

With these exercises you will practice a little bit with the ordering of operators, an important issue in quantum mechanics in general, and in quantum optics in particular. You will learn in the way some useful tricks.

- (2 points) Let's consider two operators A and B. You will first show that

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots.$$

  In order to do that, consider the function $F(\theta) = e^{\theta A} B e^{-\theta A}$, and perform its Taylor expansion: $F(\theta) = F(0) + \frac{dF}{d\theta}|_{\theta=0}\theta + \frac{1}{2!}\frac{d^2F}{d\theta^2}|_{\theta=0}\theta^2 + \dots$. From the form of $F(\theta = 1)$ you will demonstrate the expression above.

- (3 points) Consider now that $[A,[A,B]] = [B,[A,B]] = 0$, you will demonstrate now the Baker-Hausdorff formula:

$$e^{A+B} = e^A e^B e^{-[A,B]/2}.$$

  We define the operator $F(\theta) = e^{\theta(A+B)}$, and use the ansatz $F(\theta) = e^{p(\theta)A} e^{q(\theta)B} e^{r(\theta)[A,B]}$. We want to find the functions p, q and r. To find them, evaluate $dF/d\theta$ and express it in the form $dF/d\theta = GF(\theta)$, where G is an operator. Find the operator G for both the form $e^{\theta(A+B)}$ and the ansatz (for this you will need the property that you just demonstrated in the first part of the exercise). We impose that these two G operators must be the same. By equating the coefficients for A, B and $[A,B]$ you will find very simple equations for p, q and r. Once you have found these functions, just take $\theta = 1$ and you will have the Baker-Hausdorff formula.

- (5 points) This trick may be employed for more involved expressions. Consider the operators K_+, K_-, and K_3, such that $[K_+, K_-] = -2K_3$, and $[K_3, K_\pm] = \pm K_\pm$. We are interested in showing that

$$e^{\theta(K_+ - K_-)} = e^{\tanh(\theta)K_+} e^{-\ln(\cosh^2\theta)K_3} e^{-\tanh(\theta)K_-}.$$

  In order to do so, we will proceed as above. We define $F(\theta) = e^{\theta(K_+ - K_-)}$, and consider the ansatz $F(\theta) = e^{p(\theta)K_+} e^{q(\theta)K_3} e^{-p(\theta)K_-}$. As above you calculate $dF/d\theta = GF(\theta)$ for both expressions, and equate the coefficients of K_+, K_- and K_3. In this way you will get equations for p and q, and you will demonstrate the expression above.

- (1 point) The previous expression can be employed to express the squeezing operator in normal order, which is particularly useful in some occasions:

$$S(\epsilon) = e^{\epsilon^* a^2/2 - \epsilon(a^\dagger)^2/2} = (\cosh r)^{-1/2} e^{-\Gamma(a^\dagger)^2/2} e^{-\ln(\cosh r)a^\dagger a} e^{\Gamma^* a^2/2},$$

  with $\epsilon = re^{i2\phi}$, and $\Gamma = e^{2i\phi}\tanh r$. You may demonstrate this by taking $K_- = e^{-2i\phi}a^2/2$, $K_+ = K_-^\dagger$. Find K_3, and show that $K_\pm$ and K_3 fulfill the commutations rules mentioned above. Using the expression above you will find the normal ordering form of the squeezing operator.