
Exercise 1: Three-level atom (9 points)

In the class we had a look to the simplest model for the atom-light interaction, the so-called two-level atom. There we had a single electromagnetic mode, and just two levels. In this exercise you will practice a little bit with these ideas, but with a slightly more complicated system, namely a three-level atom. In this case one has three electronic levels: $\{|1\rangle, |2\rangle, |3\rangle\}$, with energies E_j, and two electromagnetic modes of frequencies ω_A and ω_B, which are coupled, respectively, to the transitions $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$. The coupling constant for the coupling $|j\rangle \leftrightarrow |3\rangle$ is g_{j3} with $j = 1, 2$ (assume both g_{j3} as real).

- (1 points) We have three electronic levels with operators $\hat{b}_{j=1,2,3}$, and two electromagnetic modes, with operators $\hat{a}_A$ and $\hat{a}_B$. Write the Hamiltonian of the system in rotating wave approximation. In order to do so, think, as we did in the class, which kind of processes are possible, and which processes violate by far energy conservation. Neglect the latter, and you will get the desired Hamiltonian.

- (1 point) Show that the states $\{|n_A, n_B, 3\rangle, |n_A + 1, n_B, 1\rangle, |n_A, n_B + 1, 2\rangle\}$ form a closed family of states, i.e. that when you apply the Hamiltonian to any one of them you get a combination of other members of the family. The notation $|n_A, n_B, j\rangle$ means that the state has n_A photons in the electromagnetic mode A, n_B photons in the mode B, and the electron is in the level $|j\rangle$.

- (2 points) Let $|\psi(t)\rangle = \psi_3(t)|n_A, n_B, 3\rangle + \psi_1(t)|n_A + 1, n_B, 1\rangle + \psi_2(t)|n_A, n_B + 1, 2\rangle$. Write the equations for $\dot{\psi}_j(t)$, with $j = 1, 2, 3$. In order to do so, employ the Schrödinger equation $i\hbar\partial_t|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$, and collect the terms in each one of the three states. As we did in the theory class for the two-level atom, remove the overall energy $E_3 + \hbar(\omega_A + \omega_B)$, and use the definition of the detunings $\Delta_{13} = \omega_A - (E_3 - E_1)/\hbar$, and $\Delta_{23} = \omega_B - (E_3 - E_2)/\hbar$.

- (3 points) Consider $\Delta_{13} = \Delta_{23} = \Delta$. As in the theory class we introduce the Rabi frequencies $\Omega_1 = g_{13}\sqrt{n_A + 1}$ and $\Omega_2 = g_{23}\sqrt{n_B + 1}$. Show that the state

  $$|D\rangle = \frac{\Omega_2|n_A + 1, n_B, 1\rangle - \Omega_1|n_A, n_B + 1, 2\rangle}{\sqrt{\Omega_1^2 + \Omega_2^2}}$$

  decouples from the rest of the states, i.e. there is no transfer between $|D\rangle$ and the rest of the states. This state $|D\rangle$ is an example of a so-called dark state, a very important idea in quantum optics.

  In order to do this exercise, take the equations for $d\psi_j/dt$, and find the combination ψ_D of ψ_1 and ψ_2 that decouples from ψ_3. You can find also the combination ψ_B that couples with ψ_3. Re-express $|\psi(t)\rangle$ as $\psi_B(t)|B\rangle + \psi_D(t)|D\rangle + \psi_3(t)|3\rangle$, and you will find the expression above. (Note: the state $|3\rangle = |n_A, n_B, 3\rangle$, and $|B\rangle$ is the state that is coupled with $|3\rangle$, the so-called bright state).

- (2 points) Obtain the state of the system $|\psi(t)\rangle$ for a given initial state $|\psi(0)\rangle = |n_A + 1, n_B, 1\rangle$. What is the probability to find the atom after some time t in $|3\rangle$?