In the lecture we had a look to the two-slit experiment, the simplest interferometer. Here you will analyze another, quite experimentally relevant, type of interferometer, the so-called *Mach-Zehnder interferometer.*

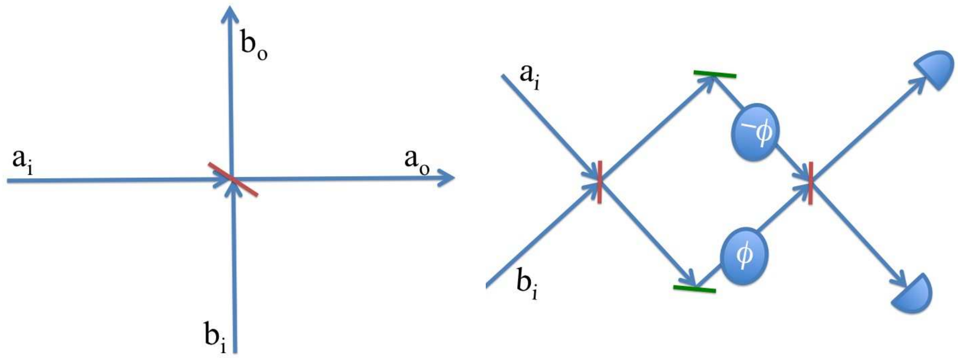

Figure 1: (left) Beam splitter; (right) Mach-Zehnder interferometer.

In the figure at the right you have a sketch of such an interferometer. One starts with two arms described by two incoming modes a_i and b_i, which go through a (typically 50/50) beam splitter, then they go along separate paths, getting a phase difference 2ϕ, and they are finally recombined in a second beam splitter. One detects the out-going modes that contain information about the phase ϕ. This phase may be induced by many different physical reasons. Measuring this phase gives hence information about physical effects. Interferometers constitute hence a basic physics tool.

In this exercise you will learn a little bit about beam splitters, Mach-Zender interferometers, and their mathematical description.

Exercise 1: Beam splitters. Hong-Ou-Mandel interference (2 points)

Let us first have a look to the idea of beam splitter (left figure). In a beam splitter two incoming modes (a_i and b_i) are combined to give two outgoing modes (a_o and b_o). Mathematically a beam splitter transforms the operators $\hat{a}_i$ and $\hat{b}_i$ describing the incoming modes, into the outgoing operators $\hat{a}_o = \cos\theta\hat{a}_i - i\sin\theta\hat{b}_i$, and $\hat{b}_o = -i\sin\theta\hat{a}_i + \cos\theta\hat{b}_i$ (for $\theta = \pi/4$ one has a 50/50 beam splitter).

We introduce at this point the spin operators $\hat{S}_x \equiv \frac{1}{2}\left(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger\right)$, $\hat{S}_y \equiv -\frac{i}{2}\left(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a}\right)$, and $\hat{S}_z \equiv \frac{1}{2}\left(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right)$ (as you may easily check these operators fulfill the necessary commutation relations for spin operators). These operators will be employed later on in the exercise.

- (1 point) Show that the beam splitter transformation may be generated by the unitary operator $\hat{R}(\theta) = e^{-i2\theta\hat{S}_x}$,. (Hint: You have to show that $\hat{R}(\theta)^\dagger\hat{a}_i R(\theta) = \cos\theta\hat{a}_i - i\sin\theta\hat{b}_i$. For showing this it may be quite useful to use the result you demonstrated in the first exercise of the first sheet).

- (1 point) Show that if the incoming state is the product number state $|1,1\rangle$ (i.e. with one photon in each mode a_i and b_i), the state after the beam splitter is of the form:
$$\cos 2\theta|1,1\rangle + i\sin 2\theta\left(\frac{|2,0\rangle+|0,2\rangle}{\sqrt{2}}\right).$$
Note that for a 50/50 beam splitter the coincidence term $|1,1\rangle$ does not appear. This result is known as *Hong-Ou-Mandel interference.*

Exercise 2: Mach-Zehnder interferometer (4 points)

We now move to the description of the full interferometer. We consider the case in which between the two beam splitters the arm of mode a gets a phase ϕ, whereas the arm of mode b gets a phase $-\phi$. This means that after the evolution between the two beam splitters, $\hat{a} \to e^{-i\phi}\hat{a}$, and $\hat{b} \to e^{i\phi}\hat{b}$.

- (1 point) Show that this is generated by the operator $\hat{Q}(\phi) = e^{-i2\phi\hat{S}_z}$, i.e. that $\hat{Q}(\phi)^\dagger\hat{a}\hat{Q}(\phi) = e^{-i\phi}\hat{a}$, and $\hat{Q}(\phi)^\dagger\hat{b}\hat{Q}(\phi) = e^{i\phi}\hat{b}$.

The final beam splitter (which we consider identical as the first one) is characterized by the transformation operator $\hat{R}(-\theta)$. The complete beam splitter is hence given by the transformation $\hat{T}(\theta,\phi) = \hat{R}(\theta)\hat{Q}(\phi)\hat{R}(-\theta)$. Such that $\hat{a} \to \hat{T}(\theta,\phi)^\dagger\hat{a}\hat{T}(\theta,\phi)$.

- (2 points) Find the general transformation, i.e. express the final outgoing operators $\{\hat{a}_f, \hat{b}_f\}$ in terms of the input ones $\{\hat{a}_i, \hat{b}_i\}$.
- (1 point) Show that for 50/50 beam splitters $\hat{T}(\phi) = e^{-2i\phi\hat{S}_y}$.

Exercise 3: Output of a Mach-Zehnder interferometer depending on the input state (4 points)

Now that you know the transformation describing the interferometer you can transform any input state (suppose in both exercises that we have 50/50 beam splitters).

- (2 points) Let us consider as input state, a coherent state $|\alpha\rangle$ in mode a_i and a coherent state $|\beta\rangle$ in mode b_i. Let $\alpha = \sqrt{\bar{n}}e^{i\varphi_\alpha}$, and $\beta = \sqrt{\bar{n}}e^{i\varphi_\beta}$. Show that
$$\sin 2\phi = \frac{\langle\hat{S}_z^f\rangle}{\bar{n}\cos(\varphi_\beta-\varphi_\alpha)},$$
with $\hat{S}_z^f \equiv \frac{1}{2}\left(\hat{a}_f^\dagger\hat{a}_f - \hat{b}_f^\dagger\hat{b}_f\right)$. Hence we may estimate the phase ϕ if we know the result of the detection after the interferometer.
- (2 points) Suppose that you enter with an input $|N,0\rangle$, i.e. purely in the a_i mode. What is the probability of detecting n photons in the arm a at the output of the interferometer? (Hint: At some point you will need to use the binomial formula).