In this exercise sheet you will practice a bit with the concept of master equation.

Exercise 1: Damped harmonic oscillator

- (2 Points) Show that the thermal density operator:

$$\hat{\rho}_T = e^{-\hbar\omega_0 \hat{a}^\dagger \hat{a}/k_B T}\left(1 - e^{-\hbar\omega_0/k_B T}\right)$$

  is a stationary solution (i.e. $d\hat{\rho}_T/dt = 0$) of the master equation of the damped harmonic oscillator of frequency ω_0 that we have discussed in the theory class.

- (1 point) Show also that at any time it is fulfilled that $\frac{d}{dt}\mathrm{Tr}\{\hat{\rho}\} = 0$. What does this mean physically?

- (2 points) In the class we assume that the modes of the reservoir followed a thermal statistics. Suppose that now we consider a vaccuum statistics, i.e. that $\langle \hat{r}_j^\dagger \hat{r}_k \rangle = 0$, and $\langle \hat{r}_j \hat{r}_k^\dagger \rangle = \delta_{j,k}$ (using the same notation as in the theory class). Obtain the new form of the Master Equation.

- (1 point) Take the master equation of the previous exercise, and consider the interaction picture (so just remove the $\hbar(\omega_0 + \Delta)$ term, i.e. the commutator term). Suppose that we consider now that the damped oscillator is driven by a resonant linear force, i.e. the system Hamiltonian has an extra term $E_0 x$. Show that the master equation becomes of the form (you just have to justify where the first term comes from since the second term should be already familiar to you!):

$$\frac{d}{dt}\hat{\rho} = i\epsilon[\hat{a} + \hat{a}^\dagger, \hat{\rho}] + \frac{\gamma}{2}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)$$

- (2 points) Calculate for the previous exercise (always in the interaction picture) $\frac{d\langle \hat{a} \rangle}{dt}$, and obtain the time evolution of $\langle \hat{a} \rangle(t)$. You will see that contrary to the un-driven case, now $\langle \hat{a} \rangle$ does not vanish for long times $t \gg 1/\gamma$.

- (2 points) Show that for the driven case, you get that the coherent state $|2i\epsilon/\gamma\rangle$ is the stationary solution (always in the interaction picture).