In this exercise sheet you will further practice the concept of master equation, learning at the same time important concepts as population trapping, adiabatic elimination and the Zeno-like effect.

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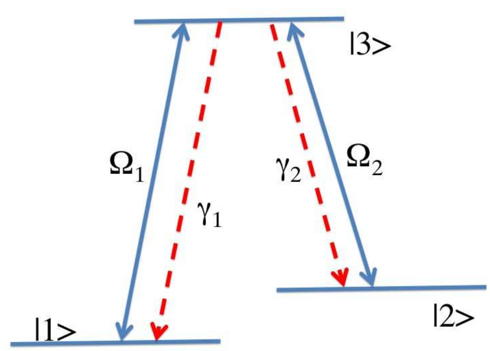
Exercise 1: Three level system (11 points)

Consider a three level system as in the figure, with an excited state $|3\rangle$ and two ground-state levels $|1\rangle$ and $|2\rangle$. It is similar to the one you already saw in the second sheet. Now we consider that there is spontaneous emission from $|3\rangle$ into both $|1\rangle$ and $|2\rangle$. For simplicity we consider both spontaneous emission rates equal, $\gamma_1 = \gamma_2 = \gamma$. We introduce the annihilation operators $\hat{a}_{j=1,2,3}$ associated to each level.



In absence of spontaneous emission, the physics is given by the Hamiltonian:

$$\hat{H}_0 = \sum_{j=1,2,3} E_j \hat{a}_j^\dagger \hat{a}_j - \sum_{j=1,2} \hbar\Omega_j \left[e^{-i\omega_{Lj}t} \hat{a}_3^\dagger \hat{a}_j + \text{H.c.} \right]$$

where ω_{Lj} is the frequency of the laser connecting $|j = 1, 2\rangle$ and $|3\rangle$, Ω_j are the corresponding Rabi frequencies, and E_j is the energy of the $j = 1, 2, 3$ level. You may take $E_3 = 0$, and hence $E_{j=1,2} = -\hbar\omega_{j3}$, with ω_{j3} the transition frequency between the states $|j = 1, 2\rangle$ and $|3\rangle$. We introduce the detunings $\Delta_{j=1,2} = \omega_{Lj} - \omega_{j3}$. We will assume that $\Delta_1 = \Delta_2 = \Delta$.

- (1 Point) Write down the Hamiltonian $\hat{H}_0$ in the interaction picture with respect to the Hamiltonian $\hat{H}_A = \sum_j E_j \hat{a}_j^\dagger \hat{a}_j$. Re-express the Hamiltonian as a function of the operators $\hat{a}_B$ and $\hat{a}_D$ of the bright and the dark states. (Note: recall from sheet 2 the definition of dark and bright state; this will allow you to write easily $\hat{a}_B$ and $\hat{a}_D$ as a function of $\hat{a}_1$ and $\hat{a}_2$, and viceversa.)

- (1 point) We will now introduce the spontaneous emission. Write the corresponding master equation in the interaction picture with $\hat{H}_A$. You do not need to derive it from scratch. Just proceed in exactly the same way as we did in the theory class on resonance fluorescence. The only difference is that now you will have two dissipative terms, one from $|3\rangle$ to $|1\rangle$ and the other from $|3\rangle$ to $|2\rangle$. (Note: as in the discussion of resonance fluorescence in the class we consider a vaccuum electromagnetic field, i.e. no thermal term in the master equation.)

- (2 points) Re-express the master equation in terms of the operators of the bright and dark states that you got in the first point.

  If you did all properly you should get $\dot{\rho} = \mathcal{L}_B\hat{\rho} + \mathcal{L}_D\hat{\rho}$, with

  $$\mathcal{L}_B = -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] + \frac{\gamma}{2}\left[2\hat{\sigma}_B^-\hat{\rho}\hat{\sigma}_B^+ - \hat{\sigma}_B^+\hat{\sigma}_B^-\hat{\rho} - \hat{\rho}\hat{\sigma}_B^+\hat{\sigma}_B^-\right]$$

  and

  $$\mathcal{L}_D = \frac{\gamma}{2}\left[2\hat{\sigma}_D^-\hat{\rho}\hat{\sigma}_D^+ - \hat{\sigma}_D^+\hat{\sigma}_D^-\hat{\rho} - \hat{\rho}\hat{\sigma}_D^+\hat{\sigma}_D^-\right]$$

  with $\hat{\sigma}_{B,D}^+ = 2\hat{a}_3^\dagger\hat{a}_{B,D}$.

- (2 points) Now we may evaluate the evolution of the populations $\rho_{jj}(t) = \langle\hat{a}_j^\dagger\hat{a}_j\rangle$ for $j = D, B, 3$ (note that $\sum_{j=D,B,3}\rho_{jj} = 1$), as well as the evolution of the correlation functions $\rho_{jk}(t) = \langle\hat{a}_j^\dagger\hat{a}_k\rangle$ (which are complex numbers, $\rho_{jk} = \rho_{jk,r} + i\rho_{jk,i}$, and $\rho_{jk} = \rho_{kj}^*$). Consider for simplicity $\Delta = 0$. Show that you may write the following closed system of equations:

  $$\begin{aligned}\frac{d}{dt}\rho_{DD} &= \gamma\rho_{33}, \\ \frac{d}{dt}\rho_{33} &= -2\gamma\rho_{33} + \Omega\rho_{3B,i} \\ \frac{d}{dt}\rho_{3B,i} &= -\gamma\rho_{3B,i} + \frac{\Omega}{2}(1 - \rho_{DD} - 2\rho_{33})\end{aligned}$$

  You see that the population in the dark state just can grow. This is because it receives population from the spontaneous emission, but what falls in the dark state remains there, since the dark state is disconnected from $|3\rangle$. This important effect is called *coherent population trapping*.

- (3 points) Try to solve the equations numerically. Consider as initial condition the state $|1\rangle$. If you assume for simplicity $\Omega_1 = \Omega_2$, you will see that this means $\rho_{33}(0) = \rho_{3B,i}(0) = 0$, and $\rho_{DD} = 1/2$. You may use the following Mathematica code (where we use a dimensionless time $t \equiv \Omega t$, and $\gamma \equiv \gamma/\Omega$):

```
tmax = 20.;
gamma = 1;
sol = NDSolve[{ ri'[t] == -gamma*ri[t] - n3[t] + 1./2.*(1 - nd[t]),
n3'[t] == ri[t] - 2*gamma*n3[t],
nd'[t] == gamma*n3[t], ri[0] == 0., n3[0] == 0.,
nd[0] == 1./2. }, { ri[t], n3[t], nd[t] }, { t, 0, tmax } ];
Plot[Evaluate[nd[t] /. sol], { t, 0, tmax }, PlotRange -> All]
```

  This program plots the evolution of $\rho_{DD}(t)$. Play a bit with the value of γ/Ω. You will see that when you increase this ratio, first the evolution becomes faster, but for a sufficiently large ratio the evolution becomes actually slowlier and slowlier. This effect is linked to the *Zeno effect* I mentioned in one of the theory classes.

- (2 points) We will finally consider the case in which $\gamma \gg \Omega$. This regime may be studied analytically using the so-called *adibatic elimination*. The damping is so fast that we can basically assume that n_3 and ρ_I are basically in a stationary regime, i.e. we may approximate in the equations above $\dot{\rho}_{33} = \dot{\rho}_{3B,i} = 0$ (you can check in the numerics that if you take a large γ, ρ_{33} and $\rho_{3B,i}$ basically are time independent). If you do so, you can eliminate ρ_{33} and $\rho_{3B,i}$, and get a simple equation for ρ_{DD}.

  If you did it properly you should get that $\rho_{DD}(t) \simeq 1 - \frac{1}{2}e^{-\frac{\Omega^2}{4\gamma}t}$ (you may check numerically that this expression reproduces well what you get from the code above for $\gamma/\Omega > 5$). You see hence that for large γ, the time scale is given by γ/Ω^2, and hence the time scale becomes larger and larger (i.e. the dynamics becomes slowlier and slowlier) when γ grows!