Exercise 1: Phase diffusion (10 points)

In this exercise you will practice a bit the ideas of Master equation, Fokker Planck equation, and Q representation.

A simple model for phase diffusion of a harmonic oscillator is given by the master equation

$$\dot{\hat{\rho}} = \Gamma\left(2\hat{n}\hat{\rho}\hat{n} - \hat{\rho}\hat{n}^2 - \hat{n}^2\hat{\rho}\right), \tag{1}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$, and $\hat{a}$ is the bosonic operator characterizing the harmonic oscillator.

Your first goal is to transform this equation into a Fokker Planck equation for the Q representation, $Q(\alpha, \alpha^*) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle$. In order to do so, use the definition of the Q representation to show that

- (1.5 points) $\frac{1}{\pi}\langle\alpha|\hat{\rho}\hat{n}|\alpha\rangle = \alpha\left(\frac{\partial}{\partial\alpha} + \alpha^*\right)Q$
- (1.5 points) $\frac{1}{\pi}\langle\alpha|\hat{n}\hat{\rho}\hat{n}|\alpha\rangle = |\alpha|^2\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*} + |\alpha|^2 + 1\right)Q$
- (1.5 points) $\frac{1}{\pi}\langle\alpha|\hat{\rho}\hat{n}^2|\alpha\rangle = \left(\alpha^2\frac{\partial^2}{\partial\alpha^2} + (2|\alpha|^2+1)\,\alpha\frac{\partial}{\partial\alpha} + |\alpha|^2\left(|\alpha|^2+1\right)\right)Q$
- (1.5 points) Use these properties to obtain the Fokker-Planck equation for the Q representation:

$$\dot{Q} = \Gamma\left(\frac{\partial}{\partial\alpha}(\alpha Q) + \frac{\partial}{\partial\alpha^*}(\alpha^* Q) + 2\frac{\partial^2}{\partial\alpha\partial\alpha^*}(|\alpha|^2 Q) - \frac{\partial^2}{\partial\alpha^2}(\alpha^2 Q) - \frac{\partial^2}{\partial(\alpha^*)^2}((\alpha^*)^2 Q)\right) \tag{2}$$

As mentioned above the master equation above is used to described phase diffusion. In order to see that this is indeed the case, consider the polar representation of the complex number $\alpha = \sqrt{I}e^{i\theta}$, where we introduce the intensity ($I = |\alpha|$) and the phase θ.

- (3 points) Show that the Fokker-Planck equation reduces to a simple diffusion equation for the phase: $\dot{Q}(I,\theta) = \Gamma\frac{\partial^2}{\partial\theta^2}Q$.
- (1 point) Show hence that starting with a defined zero phase, for short times $\langle\theta^2\rangle(t) = 2\Gamma t$.