* QUANTIZATION OF THE ELECTROMAGNETIC FIELD

· The classical electromagnetic field obeys Maxwell equations which in absence of sources are

$$\left\{\begin{array}{l} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\times\vec{B} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t} \end{array}\right\}$$

$\vec{B} \equiv$ Magnetic field

$\vec{E} \equiv$ Electric field

$\left.\begin{array}{l}\mu_0 \equiv \text{magnetic permeability} \\ \epsilon_0 \equiv \text{electric permittivity}\end{array}\right\} \mu_0 \epsilon_0 = \dfrac{1}{c^2}$

· Using the Coulomb gauge $\Rightarrow$ $\left.\begin{array}{l}\bar{B} = \bar{\nabla}\times\bar{A} \\ \bar{E} = -\partial\bar{A}/\partial t \\ \bar{\nabla}\cdot\bar{A} = 0\end{array}\right\}$ $\bar{A} \equiv$ Vector potential

From Maxwell equations one obtains then:

$$\nabla^2 \vec{A} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}$$ $\Longrightarrow$ Wave equation for the vector potential.

The solutions of this equation are of the form:

$$\vec{A}(\vec{r},t) = \vec{A}^{(+)}(\vec{r},t) + \vec{A}^{(-)}(\vec{r},t)$$

where $\vec{A}^{(-)} = \vec{A}^{(+)*}$ and

$$\vec{A}^{(+)}(\vec{r},t) = \sum_k c_k \vec{u}_k(\vec{r}) e^{-i\omega_k t} \qquad \text{with } \omega_k > 0$$

Note: here we solve the wave equation using a discrete Fourier series, instead of a continuous Fourier Transform, this is because we assume the field restricted to a certain volume of space).

· Introducing the Fourier series in the wave equation we obtain the equation for $\vec{u}_k(\vec{r})$:

$$\left(\vec{\nabla}^2 + \frac{\omega_k^2}{c^2}\right)\vec{u}_k(\vec{r}) = 0$$

- $u(\vec{k})$ should fulfill also

$$\vec{\nabla}\cdot\vec{u}_k(\vec{r}) = 0 \quad (\text{since } \vec{\nabla}\cdot\vec{A} = 0)$$

$\leftrightarrow$ Transversality condition.

For a cubical volume of side L, and choosing periodic boundary conditions, we can easily solve for $\vec{u}_k(\vec{r})$ to find

$$\vec{u}_k(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}\vec{r}}\,\hat{e}^{(\lambda)} \equiv \text{Plane-wave solution.}$$

$\left(\underline{\text{Note}}: k_j = 2\pi\, n_j/L,\quad j = x,y,z\right)$

which fulfill the orthonormality condition

$$\int_V \vec{u}_k^*(\vec{r})\,\vec{u}_{k'}(r)\,d\vec{r} = \delta_{kk'}$$

Note that $\hat{e}^{(\lambda)}$ is the polarization vector, with $\lambda = 1,2$ which indicates the 2 possible polarizations. There are just 2 possible polarizations because due to the transversality condition $\vec{k}\cdot\hat{e}^{(\lambda)} = 0$.

The vector $\vec{k} \equiv$ wavenumber vector

$$\vec{k} = (k_x, k_y, k_z) = \frac{2\pi}{L}(n_x, n_y, n_z) \qquad n_{x,y,z} = 0, \pm 1, \pm 2, \ldots$$

(Note: now k denotes $(\vec{k},\mu)$ i.e. momentum and polarization)

Hence:

$$\vec{A} = \sum_k \left(\frac{\hbar}{2\omega_k\varepsilon_0}\right)^{1/2}\left[a_k\vec{u}_k(\vec{r})\,e^{-i\omega_k t} + a_k^*\,\vec{u}_k^*(\vec{r})\,e^{i\omega_k t}\right]$$

where the normalization factors $\left(\frac{\hbar}{2\omega_k\varepsilon_0}\right)^{1/2}$ are chosen such that the amplitudes of the modes (a_k, a_k^*) are dimensionless.

· In classical electromagnetism a_k, a_k^* are complex numbers.

The quantization of the electromagnetic field is accomplished by doing

$$\left.\begin{array}{l} a_k \longrightarrow \hat{a}_k \\ a_k^* \longrightarrow \hat{a}_k^+ \end{array}\right\} \text{ mutually adjoint operators}$$

Since photons are bosons the operators $\hat{a}_k$ and $\hat{a}_k^+$ fulfill the boson commutation rules:

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^+, \hat{a}_{k'}^+] = 0$$

$$[\hat{a}_k, \hat{a}_{k'}^+] = \delta_{kk'}$$

* The operator $\hat{a}_k$ acts as ~~the~~ the annihilation operator of a photon in the mode k, whereas $\hat{a}_k^+$ acts as the creation operator.

The quantum states of each mode may now be discussed independently of one another. The states of the entire field are then described by the tensor product of the states of the different modes.

· From $\vec{A}$ we can easily obtain $\vec{E}$ and $\vec{B}$, and hence re-write the Hamiltonian

$$H = \frac{1}{2}\int [\epsilon_0 E^2 + \frac{1}{\mu_0} B^2]\, d\vec{r}$$

for the electromagnetic field in the form

$$\hat{H} = \sum_k \hbar\omega_k (\hat{a}_k^+ \hat{a}_k + 1/2)$$

This represents the sum of the number of photons in each mode multiplied by the energy of a photon in that mode, plus $\frac{1}{2}\hbar\omega_k$ (zero-point energy of every mode).

- We will have a look now to 3 possible representations of the EM field:

- FOCK STATES

- The Hamiltonian can be written in the form $\hat{H} = \sum_k \hat{H}_k$

where
$$\hat{H}_k = \hbar\omega_k (\hat{a}_k^+ \hat{a}_k + 1/2)$$

This Hamiltonian has eigenvalues $\hbar\omega_k (n_k + 1/2)$ where $n_k = 0, 1, 2, \ldots$. The corresponding eigenstates are written in the form $|n_k\rangle$ and are known as number or Fock states.

They are hence eigenstates of the number operator $\hat{n}_k = \hat{a}_k^+ \hat{a}_k$:

$$\hat{n}_k |n_k\rangle = n_k |n_k\rangle$$

The vacuum state of a given field mode is given by

$$\hat{a}_k |0\rangle = 0 \longrightarrow \langle 0 | \hat{H}_k | 0 \rangle = \frac{\hbar\omega_k}{2} \equiv \text{zero-point oscillation}$$

The creation and annihilation operators provide as usual:

$$a_k^+ |n_k\rangle = \sqrt{n_k + 1}\ |n_k + 1\rangle$$
$$a_k |n_k\rangle = \sqrt{n_k}\ |n_k - 1\rangle$$

and hence
$$|n_k\rangle = \frac{1}{\sqrt{n_k!}} (\hat{a}_k^+)^{n_k} |0\rangle$$

So it works exactly the same as for the states of an harmonic oscillator

- The Fock states are orthonormal

$$\langle n_k | n'_k \rangle = \delta_{nn'}$$

and complet $\sum_{n_k=0}^{\infty} |n_k\rangle\langle n_k| = 1$

and hence they form a complete basis of vectors for the Hilbert space.

- Number states are particularly useful for the case of few photons, but they are not the best representation of fields with a large number of photons. We will see now better representations.

- <u>COHERENT STATES</u>

- A more appropiate description of many optical fields is achieved by employing the so-called coherent states. Contrary to the Fock states they do not possess a defined number of photons. We will see later that the coherent states are states of minimal Heisenberg uncertainty. But now, we will have a look first to the basic properties of the coherent states.

The coherent states $|\alpha\rangle$ are eigenstates of the annihilation operator $\hat{a}$:

$\boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle}$ where α is a complex number.

These states are most easily generated using the <u>unitary displacement operator</u>: $\hat{D}(\alpha) = e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}}$

* We will show now that if we define $\hat{D}(\alpha)|0\rangle = |\alpha\rangle$, then $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ ⑥

· First we should re-write $\hat{D}(\alpha)$ in a more convenient way.
We will employ the so-called Baker–Hausdorff formula.
Let $\hat{A}$ and $\hat{B}$ two operators such that $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$

Then:
$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

$\hat{A} = \alpha\hat{a}^+$; $\hat{B} = -\alpha^*\hat{a}$; $[\hat{A},\hat{B}] = -|\alpha|^2[\hat{a}^+,\hat{a}] = |\alpha|^2$

Then:
$$\hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}} = e^{\alpha\hat{a}^+} e^{-\alpha^*\hat{a}} e^{-|\alpha|^2/2}$$

Let's see briefly some properties of the displacement operator:

$\hat{A} = \alpha^*\hat{a}$; $\hat{B} = -\alpha\hat{a}^+$; $[\hat{A},\hat{B}] = -|\alpha|^2$
$$\hat{D}(\alpha)^+ = e^{\alpha^*\hat{a} - \alpha\hat{a}^+} = e^{\alpha^*\hat{a}} e^{-\alpha\hat{a}^+} e^{|\alpha|^2/2}$$

Note that
$$\begin{cases} \hat{D}(\alpha)^+ = \hat{D}(-\alpha) \\ \hat{D}(\alpha)^+\hat{D}(\alpha) = \mathbb{1} \longrightarrow \hat{D}(\alpha)^+ = \hat{D}(\alpha)^{-1} \end{cases}$$

Note also that
$$\hat{D}(\alpha)^+\hat{a}\hat{D}(\alpha) = e^{\alpha^*\hat{a}} e^{-\alpha\hat{a}^+} \hat{a}\, e^{\alpha\hat{a}^+} e^{-\alpha^*\hat{a}}$$

$$[\hat{a}, e^{\alpha\hat{a}^+}] = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}[\hat{a},(\hat{a}^+)^n] = \sum_{n=1}^{\infty} \frac{\alpha^n}{(n-1)!}(\hat{a}^+)^{n-1} = \alpha e^{\alpha\hat{a}^+}$$

Then $\hat{a}\, e^{\alpha\hat{a}^+} = e^{\alpha\hat{a}^+}(\hat{a}+\alpha)$

Then: $\hat{D}(\alpha)^+\hat{a}\hat{D}(\alpha) = \hat{a}+\alpha$

* Similarly: $\hat{D}^\dagger(\alpha)\,\hat{a}^\dagger\,\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*$

OK, let's see now that if we define $\hat{D}(\alpha)|0\rangle = |\alpha\rangle$, then $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$$\hat{D}^\dagger(\alpha)\,\hat{a}\,|\alpha\rangle = \hat{D}^\dagger(\alpha)\,\hat{a}\,\hat{D}(\alpha)|0\rangle = (\hat{a}+\alpha)|0\rangle = \alpha|0\rangle$$

Then $\hat{a}|\alpha\rangle = \underbrace{\hat{D}(\alpha)\,\hat{D}^\dagger(\alpha)}_{\hat{1}}\,\hat{a}|\alpha\rangle = \hat{D}(\alpha)\,\alpha|0\rangle = \alpha|\alpha\rangle$

Hence, as we wanted to show $\boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle}$

* From this expression we can easily relate the coherent and the Fock states:

$$\left.\begin{array}{l}\langle n|\hat{a}|\alpha\rangle = \alpha\langle n|\alpha\rangle \\ \quad\; \| \\ \sqrt{n+1}\,\langle n+1|\alpha\rangle\end{array}\right\}\quad \langle n+1|\alpha\rangle = \frac{\alpha}{\sqrt{n+1}}\langle 0|\alpha\rangle$$

After recursion: $\langle n|\alpha\rangle = \dfrac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle$

Then:

$$|\alpha\rangle = \sum_n \langle n|\alpha\rangle\,|n\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle\,|n\rangle$$

and $\langle 0|\alpha\rangle = \langle 0|\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}$

Then: $\boxed{|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle} \Longrightarrow$ RELATION BETWEEN FOCK AND COHERENT STATES

Hence $|\langle\alpha|\alpha\rangle|^2 = e^{-|\alpha|^2}\sum_n \dfrac{|\alpha|^{2n}}{n!} = e^{-|\alpha|^2}e^{|\alpha|^2} = 1$

$\longrightarrow$ The coherent states are normalized.

* The probability to have n photons in a coherent state is

$$P_\alpha(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}$$

This is a so-called <u>POISSONIAN DISTRIBUTION</u> (see pages 8' and 8'')

Although as we have seen, coherent states are normalized to 1, it is not the case that two different coherent states are orthogonal. Let's see this:

$$\langle\beta|\alpha\rangle = \langle 0|\hat{D}^\dagger(\beta)\hat{D}(\alpha)|0\rangle$$

$$= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\langle 0|e^{-\beta\hat{a}^\dagger} e^{\beta^*\hat{a}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}|0\rangle$$

$$= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \underbrace{\langle 0|e^{\beta^*\hat{a}}}_{\sum_{n'=0}^{\infty}\frac{(\beta^*)^{n'}}{n'!}\langle 0|(\hat{a})^{n'}} \underbrace{e^{\alpha\hat{a}^\dagger}|0\rangle}_{\sum_{n=0}^{\infty}\frac{(\alpha)^n}{n!}(\hat{a}^\dagger)^n|0\rangle}$$

$$\underbrace{\sum_{n=0}^{\infty}\frac{(\alpha\beta^*)^n}{n!}\underbrace{\frac{\langle 0|(\hat{a})^n}{\sqrt{n!}}\frac{(\hat{a}^\dagger)^n|0\rangle}{\sqrt{n!}}}_{\langle n|n\rangle = 1}}_{e^{\alpha\beta^*}}$$

Then: $\langle\beta|\alpha\rangle = e^{-\frac{1}{2}[|\alpha|^2+|\beta|^2]} e^{\alpha\beta^*}$

$$\boxed{|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}}$$

$\rightarrow$ Two different coherent states are not orthogonal, although if $|\alpha-\beta| \gg 1$, $|\langle\beta|\alpha\rangle| \simeq 0$

* SOME PROPERTIES OF THE POISSONIAN DISTRIBUTION

* $$P_\alpha(n) = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}$$

: let's calculate the mean number of photons

$$\langle n\rangle = \sum_{n=0}^{\infty} n\, P_\alpha(n) = \sum_{n=0}^{\infty} n \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} = e^{-|\alpha|^2} |\alpha|^2 \sum_{n=1}^{\infty} \frac{|\alpha|^{2(n-1)}}{(n-1)!}$$

$$= e^{-|\alpha|^2} |\alpha|^2 e^{+|\alpha|^2} = |\alpha|^2$$

This was expected because the expected value of $\hat{a}^\dagger \hat{a}$ for a coherent state is:

$$\langle n\rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = \langle \alpha | \alpha^* \alpha | \alpha \rangle = |\alpha|^2$$

Let's calculate $(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$

$\langle n^2 \rangle$ can be easily evaluated as follows:

$$\frac{d}{d|\alpha|^2} \langle n \rangle = 1$$

$$= \frac{d}{d|\alpha|^2} \sum_{n=0}^{\infty} n\, P_\alpha(n) = \sum_{n=0}^{\infty} n \frac{dP_\alpha(n)}{d|\alpha|^2} =$$

$$= \sum_{n=0}^{\infty} n \left[\frac{n (|\alpha|^2)^{(n-1)}}{n!} e^{-|\alpha|^2} \right] - \sum_{n=0}^{\infty} n \left[\frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \right]$$

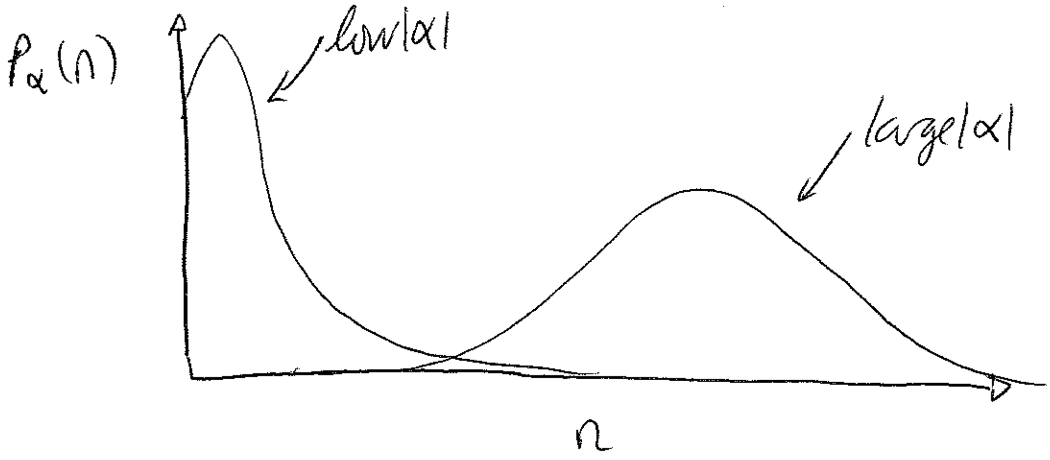
$$= \frac{1}{|\alpha|^2} \sum_{n=0}^{\infty} n^2 P_\alpha(n) - \sum_{n=0}^{\infty} n\, P_\alpha(n) = \frac{1}{|\alpha|^2} \langle n^2 \rangle - \langle n \rangle$$

Then $\langle n^2 \rangle = \langle n \rangle [\langle n \rangle + 1]$

and hence $\boxed{(\Delta n)^2 = \langle n \rangle}$ — In a Poissonian distribution the square of the variance is equal to the average.

* A distribution is called <u>subpoissonian</u> if $(\Delta n)^2 < \langle n \rangle$ and <u>superpoissonian</u> if $(\Delta n)^2 > \langle n \rangle$.

• graphically, a Poissonian distribution looks like this:



• For large $|\alpha|$ it converges to a gaussian distribution

$$\frac{e^{-(n-\bar{n})^2/2\bar{n}}}{\sqrt{2\bar{n}\pi}}$$

* The coherent state form an overcomplete set that spans a 2D continuum of states (2D because $\alpha, \alpha^* \equiv$ complex plane)

- One may easily prove that

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha = 1$$

PROOF:

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{|n\rangle\langle m|}{\sqrt{n!\,m!}}\int e^{-|\alpha|^2}\alpha^{*m}\alpha^n\, d^2\alpha \overset{\alpha = re^{i\theta}}{=}$$

$$= \frac{1}{\pi}\sum_{n,m=0}^{\infty}\frac{|n\rangle\langle m|}{\sqrt{n!\,m!}}\int_0^{\infty} r\,dr\; e^{-r^2}\, r^{n+m}\underbrace{\int_0^{2\pi} d\theta\, e^{i(n-m)\theta}}_{2\pi\delta_{n,m}} \overset{\varepsilon = r^2}{=}$$

$$= \sum_{n=0}^{\infty}\frac{|n\rangle\langle n|}{n!}\underbrace{\int_0^{\infty} d\varepsilon\, e^{-\varepsilon}\,\varepsilon^n}_{n!} = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1$$

- So summarizing:

$$\left.\begin{aligned}
&\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\\
&|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\\
&|\langle\alpha|\alpha\rangle|^2 = 1\\
&|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}\\
&\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \pi
\end{aligned}\right\}$$

* Coherent states are very useful to describe lasers, and they are employed for expanding optical fields in laser physics and non-linear optics.

* We will come back later on in these lectures to the idea of coherent states.

• SQUEEZED STATES

* We have already seen that the operators $\hat{a}$ and $\hat{a}^+$ for a given mode are exactly similar to the ladder operators of an harmonic oscillator.

* Remember that for an harmonic oscillator one can link the operators $\hat{a}$ and $\hat{a}^+$ with the operators position ($\hat{q}$) and momentum ($\vec{p}$)

$$\left.\begin{aligned}\hat{q} &= \sqrt{\frac{\hbar}{2\omega}}\,(\hat{a}+\hat{a}^+)\\ \hat{p} &= i\sqrt{\frac{\hbar\omega}{2}}\,(\hat{a}-\hat{a}^+)\end{aligned}\right\} \quad (\text{let } m=1)$$

• Let's consider a coherent state $|\alpha\rangle$

$$\langle\hat{q}\rangle = \sqrt{\frac{\hbar}{m\omega}}\,\langle\alpha|\hat{a}+\hat{a}^+|\alpha\rangle = \sqrt{\frac{\hbar}{2\omega}}(\alpha+\alpha^*)$$

$$\langle\hat{q}^2\rangle = \frac{\hbar}{2\omega}\langle\alpha|\hat{a}^2+\hat{a}^{+2}+\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}|\alpha\rangle = \frac{\hbar}{2\omega}\left(1+(\alpha+\alpha^*)^2\right)$$

$$(\Delta q)^2 = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2 = \frac{\hbar}{2\omega}$$

Similarly $(\Delta p)^2 = \frac{\hbar\omega}{2}$

Then, for a coherent state $\Delta q\,\Delta p = \frac{\hbar}{2}$, i.e. the product of the uncertainties is a minimum (remember the Heisenberg uncertainty principle).

• Let's define the more convenient operators $\hat{X}_1$ and $\hat{X}_2$:

$$\left.\begin{aligned}\hat{a} &= \tfrac{1}{2}(\hat{X}_1+i\hat{X}_2)\\ \hat{a}^+ &= \tfrac{1}{2}(\hat{X}_1-i\hat{X}_2)\end{aligned}\right\} \quad \left(\text{i.e. } \hat{X}_1 \equiv \sqrt{\frac{2\omega}{\hbar}}\,\hat{q}\,,\ \hat{X}_2 \equiv -\sqrt{\frac{2}{\hbar\omega}}\,\hat{p}\right)$$

$$\text{i.e. } [\hat{X}_1,\hat{X}_2] = 2i$$

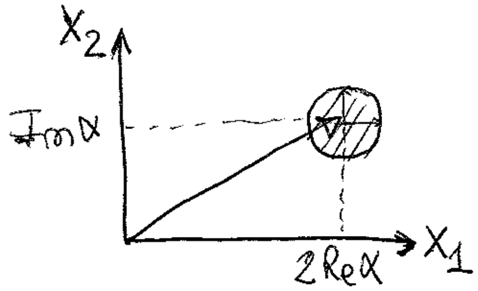
$$\Delta\hat{X}_1\cdot\Delta\hat{X}_2 \geq 1$$

(X_1 and X_2 are called quadratures)

· For a coherent state, as we just saw $\Delta X_1 = 1$, $\Delta X_2 = 1$ and hence $\Delta X_1 \Delta X_2 = 1$ (minimal uncertainty).

· The variables X_1 and X_2 are the real and imaginary part of ~~α~~ complex numbers, and may be represented in a two-dimensional graph.

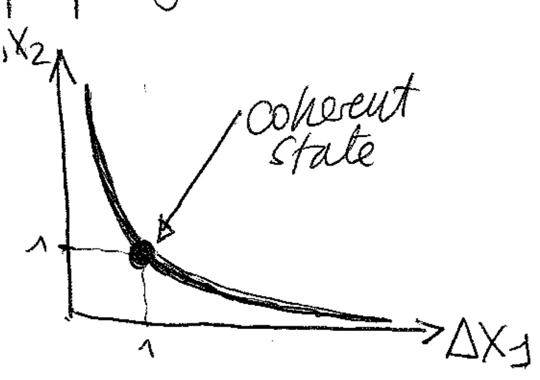
A coherent state $|\alpha\rangle$ may be represented as a circle like in the figure. The center of the circle lies at the average $\frac{1}{2}\langle X_1 + iX_2\rangle = \alpha$ and it has a radius $\Delta X_1 = \Delta X_2 = 1$ (which account for uncertainties)



· The coherent states are only a particular example of states with minimal uncertainty

$$\Delta X_1 \, \Delta X_2 = 1$$

Obviously all states such that $\Delta X_2 = \frac{1}{\Delta X_1}$ fulfill the same property. If we plot ΔX_2 vs ΔX_1 these states form a hyperbola. Only points at the right of the hyperbola fulfill $\Delta X_2 \geq 1/\Delta X_1$ and hence the uncertainty principle (and thus they are physical).



· It's important to notice that we can have $\Delta X_1 < 1$ or $\Delta X_2 < 1$, but we have to pay the price that $\Delta X_2 > 1$ ~~and~~ or $\Delta X_1 > 1$, i.e. we can gain certainty in one quadrature but we loose it in the other.

- These states are called <u>squeezed states</u>.
- The squeezed states can be "generated" mathematically by using the <u>Squeeze operator</u>:

$$\hat{S}(\epsilon) = e^{\frac{1}{2}\epsilon^* \hat{a}^2 - \frac{1}{2}\epsilon \hat{a}^{+2}}$$

where ϵ is a complex number $\epsilon = re^{2i\phi}$

- The squeeze operator fulfills some useful properties

  * $\hat{S}(\epsilon)^+ = \hat{S}(\epsilon)^{-1} = \hat{S}(-\epsilon)$

  * When operated on $\hat{a}$ it satisfies

$$\hat{S}(\epsilon)^+ \hat{a}\, \hat{S}(\epsilon) = \hat{a}\cosh r - \hat{a}^+ e^{2i\phi}\sinh r$$

Let's show this.

Let $\hat{a}(\lambda) = \hat{S}^+(\lambda\epsilon)\hat{a}\,\hat{S}(\lambda\epsilon)$. Clearly $\hat{a}(0) = \hat{a}$

$\hat{a}^+(\lambda) = \hat{S}^+(\lambda\epsilon)\hat{a}^+\hat{S}(\lambda\epsilon)$ $\quad\quad \hat{a}^+(0) = \hat{a}^+$

Let's differentiate:

$$\frac{d\hat{a}(\lambda)}{d\lambda} = \frac{d\hat{S}^+}{d\lambda}(\lambda\epsilon)\hat{a}\,\hat{S}(\lambda\epsilon) + \hat{S}^+(\lambda\epsilon)\,\hat{a}\frac{d}{d\lambda}\hat{S}(\lambda\epsilon)$$

$$\frac{d\hat{S}^+}{d\lambda}(\lambda\epsilon) = \frac{d}{d\lambda}e^{-\frac{1}{2}\lambda[\epsilon^*\hat{a}^2 - \epsilon\hat{a}^{+2}]} = -\hat{S}^+(\lambda\epsilon)\frac{1}{2}[\epsilon^*\hat{a}^2 - \epsilon\hat{a}^{+2}]$$

$$\frac{d\hat{S}}{d\lambda}(\lambda\epsilon) = \frac{1}{2}[\epsilon^*\hat{a}^2 - \epsilon\hat{a}^{+2}]\,\hat{S}(\lambda\epsilon)$$

Then

$$\frac{d\hat{a}(\lambda)}{d\lambda} = \hat{S}^+(\lambda\epsilon)\left[\hat{a}, \tfrac{1}{2}(\epsilon^*\hat{a}^2 - \epsilon\hat{a}^{+2})\right]\hat{S}(\lambda\epsilon)$$

$$= \hat{S}^+(\lambda\epsilon)\,(-\epsilon\hat{a}^+)\,\hat{S}(\lambda\epsilon) = -\epsilon\,\hat{a}^+(\lambda)$$

Similarly: $\frac{d\hat{a}^+}{d\lambda}(\lambda) = -\epsilon^*\hat{a}(\lambda)$

Hence: $\frac{d^2}{d\lambda^2}\hat{a}(\lambda) = |\epsilon|^2 \hat{a}(\lambda) \longrightarrow \hat{a}(\lambda) = \hat{c}_1 e^{|\epsilon|\lambda} + \hat{c}_2 e^{-|\epsilon|\lambda}$

Let's determine $\hat{c}_1$ and $\hat{c}_2$:

- $\hat{a}(0) = \hat{a} \longrightarrow \hat{c}_1 + \hat{c}_2 = \hat{a}$
- $\left[\frac{d}{d\lambda}\hat{a}(\lambda)\right]_{\lambda=1} = -\epsilon \hat{a}^+ = |\epsilon|(\hat{c}_1 - \hat{c}_2) \rightarrow \hat{c}_1 - \hat{c}_2 = -\frac{\epsilon}{|\epsilon|}\hat{a}^+ = -e^{2i\phi}\hat{a}^+$

Then: $\hat{c}_1 = \frac{1}{2}(\hat{a} - e^{2i\phi}\hat{a}^+)$; $\hat{c}_2 = \frac{1}{2}(\hat{a} + e^{-2i\phi}\hat{a}^+)$; and then:

$\hat{a}(\lambda) = \frac{1}{2}(\hat{a} - e^{2i\phi}\hat{a}^+)e^{|\epsilon|\lambda} + \frac{1}{2}(\hat{a} + e^{2i\phi}\hat{a}^+)e^{-|\epsilon|\lambda}$

Hence:

$\hat{a}(1) = \frac{1}{2}(e^{|\epsilon|} + e^{-|\epsilon|})\hat{a} - e^{2i\phi}\frac{1}{2}(e^{|\epsilon|} - e^{-|\epsilon|})\hat{a}^+$

And, as we wanted

$\hat{S}(\epsilon)^+ \hat{a}\, \hat{S}(\epsilon) = \mathrm{ch}r\, \hat{a} - e^{2i\phi}\,\mathrm{sh}r\, \hat{a}^+$

Similarly

$\hat{S}(\epsilon)^+ \hat{a}^+ \hat{S}(\epsilon) = \mathrm{ch}r\, \hat{a}^+ - e^{-2i\phi}\,\mathrm{sh}r\, \hat{a}$

- Let $\left.\begin{array}{l}\hat{b} = e^{-i\phi}\hat{a} \\ \hat{b}^+ = e^{i\phi}\hat{a}^+\end{array}\right\} \Rightarrow$ rotated operators (just rotate the quadratures in the (X_1, X_2) plane)

Then: $\hat{S}(\epsilon)^+ \hat{b}\, \hat{S}(\epsilon) = \hat{b}\,\mathrm{ch}r - \hat{b}^+\mathrm{sh}r$

$\hat{S}(\epsilon)^+ \hat{b}^+ \hat{S}(\epsilon) = \hat{b}^+\mathrm{ch}r - \hat{b}\,\mathrm{sh}r$

Let's consider the quadratures $\hat{Y}_1 = \hat{b}^+ + \hat{b}$, $\hat{Y}_2 = i(\hat{b}^+ - \hat{b})$

Then: $\hat{S}(\epsilon)\hat{Y}_1 \hat{S}(\epsilon) = e^{-r}\hat{Y}_1$

$\hat{S}(\epsilon)\hat{Y}_2 \hat{S}(\epsilon) = e^{r}\hat{Y}_2$.

- Hence, the squeeze operator $\hat{S}(\epsilon)$ attenuates one component of the rotated complex amplitude, and it amplifies the other quadrature.

The degree of attenuation and amplification is determined by the <u>squeeze factor</u> r.

- The squeezed state $|\alpha, \epsilon\rangle$ is obtained as:

$$|\alpha, \epsilon\rangle = \underbrace{\hat{D}(\alpha) \underbrace{\hat{S}(\epsilon)|0\rangle}_{\text{first squeeze the vacuum}}}_{\text{then displace it}}$$

(Note: $\hat{S}(\epsilon)|0\rangle$ is the so-called squeezed vacuum)

- It's interesting to have a look to the properties of the averages and variances of $|\alpha, \epsilon\rangle$:

One can easily see (exercise) that:

- $\langle \hat{a} \rangle = \alpha$
- $\langle \hat{a}^+ \rangle = \alpha^*$
- $\langle \hat{a}^2 \rangle = \alpha^2 - e^{2i\phi} \operatorname{sh} r \operatorname{ch} r$
- $\langle \hat{a}^{+2} \rangle = \alpha^{*2} - e^{-2i\phi} \operatorname{sh} r \operatorname{ch} r$
- $\langle \hat{a}^+ \hat{a} \rangle = |\alpha|^2 + \operatorname{sh}^2 r$
- $\langle \hat{a} \hat{a}^+ \rangle = |\alpha|^2 + \operatorname{ch}^2 r$

Then, e.g.: $\langle \hat{Y}_2 \rangle = i[\langle \hat{b}^+ \rangle - \langle \hat{b} \rangle] = i[e^{i\phi}\langle \hat{a}^+ \rangle - e^{-i\phi}\langle \hat{a} \rangle]$

$$= i(e^{i\phi}\alpha^* - e^{-i\phi}\alpha)$$

$$\langle \hat{Y}_2 \rangle^2 = -[2|\alpha|^2 + e^{2i\phi}\alpha^{*2} + e^{-2i\phi}\alpha^2]$$

and $\langle \hat{Y}_2^2 \rangle = -\langle \hat{b}^{+2} + \hat{b}^2 - \hat{b}^+\hat{b} - \hat{b}\hat{b}^+ \rangle$

$$= -\{ e^{2i\phi}\langle \hat{a}^{+2}\rangle + e^{-2i\phi}\langle \hat{a}^2\rangle - \langle \hat{a}^+\hat{a}\rangle - \langle \hat{a}\hat{a}^+\rangle \}$$

$$= -e^{2i\phi}(\alpha^{*2} - e^{-2i\phi}\,\mathrm{sh}r\,\mathrm{ch}r) - e^{-2i\phi}(\alpha^2 - e^{2i\phi}\,\mathrm{sh}r\,\mathrm{ch}r)$$

$$+ |\alpha|^2 + \mathrm{sh}^2 r + |\alpha|^2 + \mathrm{ch}^2 r$$

$$= 2\,\mathrm{sh}r\,\mathrm{ch}r + \mathrm{ch}^2 r + \mathrm{sh}^2 r + 2|\alpha|^2 - e^{2i\phi}\alpha^{*2} - e^{-2i\phi}\alpha^2$$

Hence $(\Delta Y_2)^2 = \langle \hat{Y}_2^2\rangle - \langle \hat{Y}_2\rangle^2 = (\mathrm{ch}r + \mathrm{sh}r)^2 = e^{2r}$

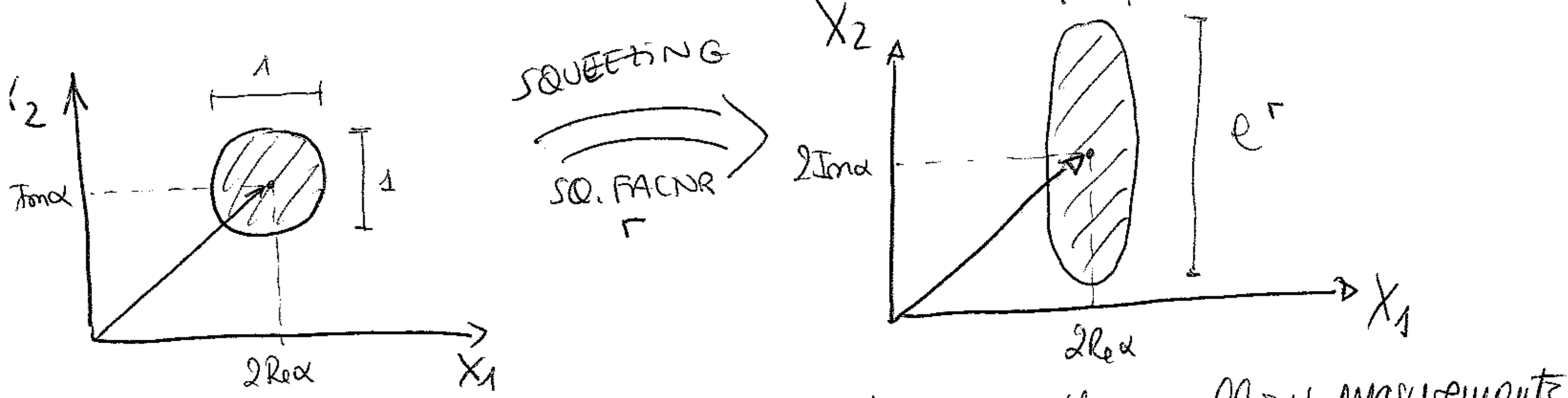
$$\boxed{\Delta Y_2 = e^{+r}}$$

Similarly $\boxed{\Delta Y_1 = e^{-r}}$

Thus, similarly as for the coherent state $\Delta Y_1 \Delta Y_2 = 1$, but now the squeezed state has unequal uncertainties.

Graphically (considering $\phi = 0$):

COHERENT STATE



The squeezed states are very important, since they allow measurements of one of the quadratures to extremely high precisions. We will meet the coherent and squeezed states many times during these lectures.

Photon number distribution in a squeezed state

* For the squeezed state $|\alpha, \epsilon\rangle$ one can calculate (exercise) that:

$$\langle n \rangle = |\alpha|^2 + sh^2 r$$

$$(\Delta n)^2 = |\alpha ch r - \alpha^* e^{2i\phi} sh r|^2 + 2 ch^2 r\, sh^2 r$$

* For $r=0$ (no squeezing) we recover $(\Delta n)^2 = \langle n \rangle = |\alpha|^2$ since we have (as we saw before) a Poissonian distribution.

When $r \neq 0$ the situation is different.

Let's consider $\phi = 0$, and α real.

* If $|\alpha| \to \infty$:

$$\langle n \rangle \simeq |\alpha|^2$$

$$(\Delta n)^2 \simeq |\alpha|^2 e^{-2r}$$

then for $r > 0 \longrightarrow (\Delta n)^2 < \langle n \rangle \Rightarrow$ subpoissonian

$r < 0 \longrightarrow (\Delta n)^2 > \langle n \rangle \Rightarrow$ superpoissonian.

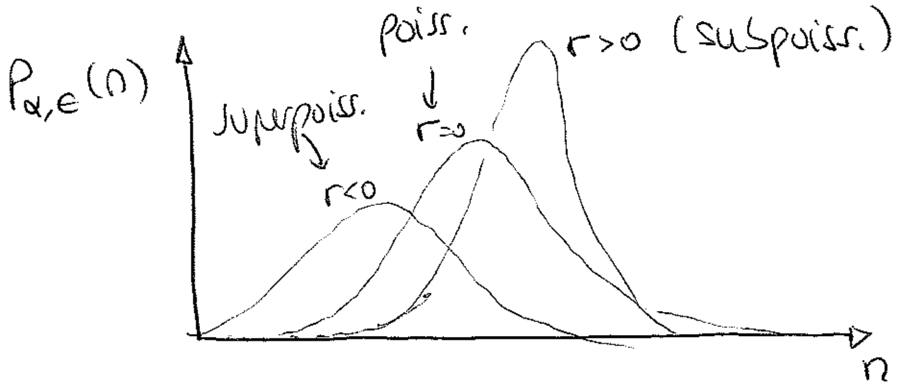
<u>Note</u>: The exact photon number distribution for a squeezed state $|\alpha, \epsilon\rangle$ is (without proof):

$$P_{\alpha,\epsilon}(n) = \frac{1}{\mu\, n!}\left(\frac{\nu}{2\mu}\right)^n \left[H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right)\right]^2 e^{-|\beta|^2 + 2\,\mathrm{Re}\left[\frac{\nu\beta^2}{2\mu}\right]}$$

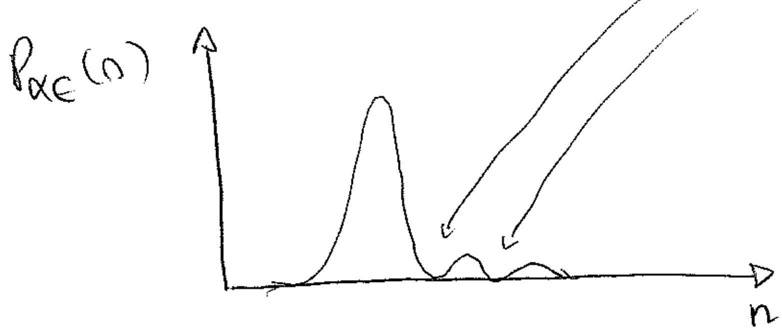
where $\nu = e^{2i\phi} sh r$ // $\beta = \mu\alpha + \nu\alpha^*$

$\mu = ch r$ // $H_n(\dots) \equiv$ Hermite polynomials

· graphically



For large $|\epsilon|$ it may even show oscillations:



These oscillations have been interpreted as interferences in phase space.

· Squeezing and the variance of the electric field

We already show that for a given mode $(\omega, \bar{k})$

$$\vec{A} = \left(\frac{\hbar}{2\omega\epsilon_0}\right)^{1/2} \frac{1}{L^{3/2}} \left[\hat{a} e^{-i(\omega t - \bar{k}\bar{r})} + \hat{a}^\dagger e^{i(\omega t - \bar{k}\bar{r})}\right]$$

Hence

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -i \left(\frac{\hbar\omega}{2\cancel{\omega}\epsilon_0}\right)^{1/2} \frac{1}{L^{3/2}} \left[\hat{a}^\dagger e^{i(\omega t - \bar{k}\bar{r})} - \hat{a} e^{-i(\omega t - \bar{k}\bar{r})}\right]$$

Since $\hat{a} = \dfrac{\hat{X}_1 + i\hat{X}_2}{2}$, $\hat{a}^\dagger = \dfrac{\hat{X}_1 - i\hat{X}_2}{2}$:

$$\vec{E} = \left(\frac{\hbar\omega}{2\epsilon_0}\right)^{1/2} \frac{1}{L^{3/2}} \left[\sin(\omega t - \bar{k}\bar{r})\,\hat{X}_1 - \cos(\omega t - \bar{k}\bar{r})\,\hat{X}_2\right].$$

We can easily calculate then the variance of $\vec{E}$:

$$(\Delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2 = \left(\frac{\hbar\omega}{2\epsilon_0 L^3}\right)\left[\sin^2(\omega t - \vec{k}\vec{r})\,\Delta X_1 + \cos^2(\omega t - \vec{k}\vec{r})\,\Delta X_2 + 2\sin(\omega t - \vec{k}\vec{r})\cos(\omega t - \vec{k}\vec{r})\,\Delta(X_1, X_2)\right]$$

where $\Delta(X_1, X_2) = \frac{1}{2}\langle X_1 X_2 + X_2 X_1 \rangle - \langle X_1 \rangle \langle X_2 \rangle$

for a state with minimal uncertainty (i.e. belonging to the subset of squeezed states) $\Rightarrow \Delta(X_1, X_2) = 0$

and:

$$(\Delta E)^2 = \left(\frac{\hbar\omega}{2\epsilon_0 L^3}\right)\left[\sin^2(\omega t - \vec{k}\vec{r})\,\Delta X_1 + \cos^2(\omega t - \vec{k}\vec{r})\,\Delta X_2\right]$$

For a coherent state the variance is a constant since $\Delta X_1 = \Delta X_2 = 1 \longrightarrow (\Delta E)^2 = \left(\frac{\hbar\omega}{2\epsilon_0 L^3}\right)$

- If $\Delta X_2 < \Delta X_1$ the fluctuations are smaller when the mean is big (i.e. for $\cos\omega t \simeq \pm 1$) $\longrightarrow$ For this case the amplitude of the oscillations is better defined than in a coherent state $\Longrightarrow$ <u>amplitude - squeezing</u>.

If $\Delta X_2 > \Delta X_1$ the fluctuations are larger when the mean is small ($\cos\omega t \simeq \pm 1$) $\longrightarrow$ The phase of the oscillations is better defined than in a coherent state $\longrightarrow$

$\longrightarrow$ <u>phase - squeezing</u>

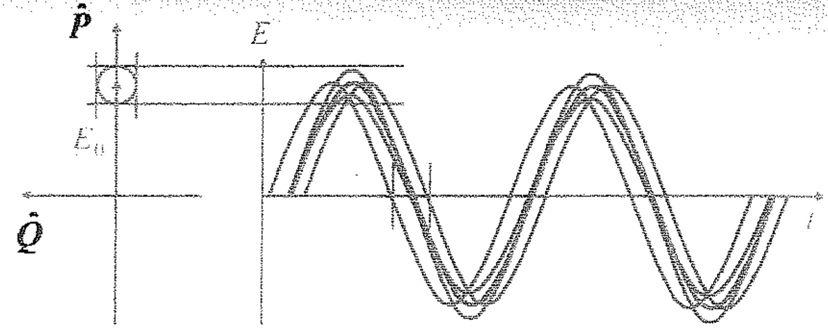


Figure 3.4.: Coherent EM field

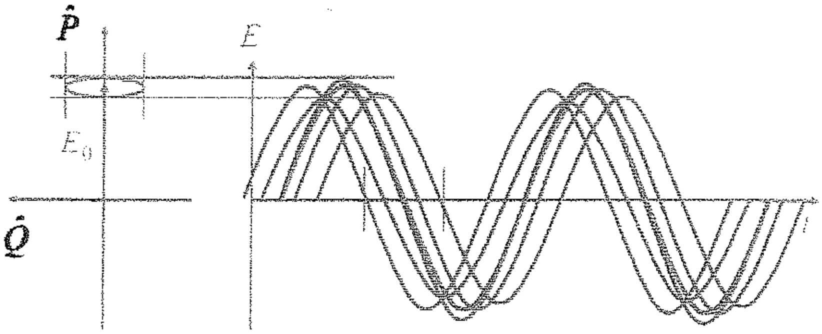


Figure 3.5.: Amplitude squeezed field

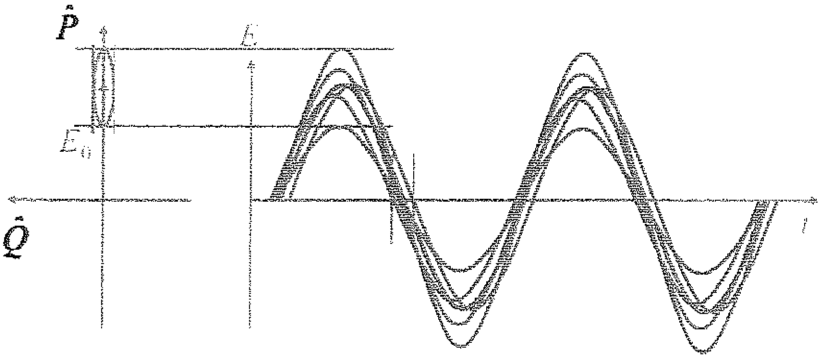


Figure 3.6.: Phase squeezed field

* Squeezed states ~~of~~ light can be generated in different ways in so-called nonlinear optical processes. We will not discuss for the moment how this is done, we will leave it for a future class.

* A <u>final remark</u>: we have introduced the concept of Fock, coherent and squeezed states for photons. One can define them as well for any other particle, e.g. electrons, atoms, etc. The idea is hence applicable <u>in general</u>!

THE PHASE OPERATOR

- Let's now briefly discuss the problem of defining an operator corresponding to the phase of the field.
- The so-called Susskind-Glogower (SG) phase operator works out the analogy with the classical polar decomposition of a complex amplitude:

$$e^{i\hat{\phi}} \equiv (\hat{a}\hat{a}^{+})^{-1/2}\,\hat{a}$$

(Note: one "divides" by $\hat{a}\hat{a}^{+}$ in order to avoid dividing by zero)

Clearly $e^{i\hat{\phi}}|n+1\rangle = |n\rangle \longrightarrow e^{i\hat{\phi}} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$

The eigenstates $|e^{i\phi}\rangle$ of the operator $e^{i\hat{\phi}}$ are

$$e^{i\hat{\phi}}|e^{i\phi}\rangle = e^{i\phi}|e^{i\phi}\rangle \qquad (\phi \text{ is defined in } [-\pi,\pi])$$

$$|e^{i\phi}\rangle \equiv \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$$

The main problem of $e^{i\hat{\phi}}$ is that it isn't Hermitian.

Clearly $[e^{i\hat{\phi}},(e^{i\hat{\phi}})^{\dagger}] = |0\rangle\langle 0|$ $\quad$ ($e^{i\hat{\phi}}$ isn't unitary).

The eigenstates are however useful, although they aren't orthogonal. They form a complete set

$$\int_{-\pi}^{\pi} |e^{i\phi}\rangle\langle e^{i\phi}|\,\frac{d\phi}{2\pi} = \hat{1}$$

This property allows to define the <u>phase probability distribution</u> for a given state $|\psi\rangle$

$$\boxed{P(\phi) = \frac{1}{2\pi}|\langle e^{i\phi}|\psi\rangle|^2}$$

which is obviously positive and normalized $\int_{-\pi}^{\pi} P(\phi)\,d\phi = 1$.

For a Fock state $|n_0\rangle$:

$$\langle e^{i\phi}|n_0\rangle = e^{in_0\phi} \longrightarrow P(\phi) = \frac{1}{2\pi}$$

Thus a Fock state has a completely undetermined phase (a consequence of having a fully determined number of photons).

For a coherent state with $\alpha = |\alpha| e^{i\phi_0}$

$$\langle e^{i\phi}/\alpha\rangle = \sum_{n=0}^{\infty} e^{-in\phi} \langle n|\alpha\rangle = \sum_{n=0}^{\infty} e^{-in\phi} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}$$

$$= \sum_{n=0}^{\infty} \frac{|\alpha|^n}{\sqrt{n!}} e^{-|\alpha|^2/2} e^{-in(\phi-\phi_0)}$$

Then $$P(\phi) = \frac{e^{-|\alpha|^2}}{2\pi} \sum_{n,n'=0}^{\infty} \frac{|\alpha|^{n+n'}}{\sqrt{n!\,n'!}} e^{-i(n-n')(\phi-\phi_0)}$$

Clearly $P(\phi)$ is an even function of $\phi - \phi_0$.

Then $\langle\phi\rangle = \int \phi\, P(\phi) = \phi_0$

* One can see that for sufficiently large $|\alpha|$

$$\langle(\phi-\phi_0)^2\rangle \simeq \frac{1}{4|\alpha|^2}$$

Then, the larger $|\alpha|$ the better the phase determination.

Pegg-Barnett (PB) phase operator

- Pegg and Burnett defined a hermitian phase operator by considering a finite subspace spanned by Fock states with 0, 1, … s photons:

$$|\phi_0\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\phi_0} |n\rangle$$

One can then form a family of orthogonal states:

$$|\phi_m\rangle = e^{i\frac{\hat{a}^\dagger \hat{a}\, m\, 2\pi}{s+1}} |\phi_0\rangle \qquad m = 0, 1, \dots s$$

with $\phi_m = \phi_0 + \frac{2\pi m}{s+1}$ (i.e. $\phi \leq \phi_m < \phi_0 + 2\pi$)

The PG phase operator is:

$$\hat{\phi} = \sum_{m=0}^{s} \phi_m |\phi_m\rangle\langle\phi_m|$$

which is Hermitian (the states ϕ_m are orthogonal).

One can then define a probability distribution for a state $|\psi\rangle$

$P_m = |\langle\phi_m|\psi\rangle|^2$ (where $|\psi\rangle$ is any vector of the truncated space $(s+1)$-dimensional).

When $s \to \infty$, $\hat{\phi}$ doesn't converge to a Hermitian operator, but P_m does converge to the SG phase distribution with $\phi = \lim_{m,s\to\infty} \phi_m$:

$$P_{SG}(\phi) \doteq \lim_{s,m\to\infty} \left[\frac{2\pi}{s+1} P_m\right]$$