* Atom–Light Interaction

- Up to now we have studied only the quantized electromagnetic field. We will analyze now its interaction with matter.
- The Hamiltonian describing the interaction between the electromagnetic field and an electron (we assume just one single (valence) electron in the atom which interacts with the field) is:

$$\hat{H} = \frac{1}{2m}(\underbrace{\hat{p} - e\hat{A}}_{\text{vector potential}}) + \underbrace{\hat{V}(r)}_{\substack{\text{Coulomb potential}\\ \text{(electron–nucleus)}}} + \underbrace{\sum_k \hbar\omega_k \hat{a}_k^+ \hat{a}_k}_{\substack{\text{Quantized electromagnetic field as we saw before}\\ \text{(we forget here the vacuum energy)}}}$$

$$= \underbrace{\frac{1}{2m}\hat{p}^2 + \hat{V}(r)}_{\hat{H}_A \equiv \text{atomic Hamiltonian}} \underbrace{- \frac{e}{2m}(\hat{p}\hat{A} + \hat{A}\hat{p}) + \frac{e^2}{2m}\hat{A}^2}_{\hat{H}_{INT} \equiv \text{interaction Hamiltonian (Atom-light)}} + \underbrace{\sum_k \hbar\omega_k \hat{a}_k^+ \hat{a}_k}_{\hat{H}_F \equiv \text{electrom. field}}$$

- Let's have a closer look to the interaction Hamiltonian.
- One can easily show that $[\hat{p}, \hat{A}] = 0$

$$[\hat{p}, \hat{A}]\psi = (\hat{p}\hat{A} - \hat{A}\hat{p})\psi = -i\hbar\vec{\nabla}[\hat{A}\psi] + \hat{A}\, i\hbar\vec{\nabla}\psi$$

$$= -i\hbar\psi(\vec{\nabla}A) - \cancel{i\hbar\hat{A}\vec{\nabla}\psi} + \cancel{i\hbar\hat{A}\vec{\nabla}\psi} = \quad \text{Coulomb gauge } \vec{\nabla}A = 0$$

$$= 0$$

then $\hat{H}_{INT} = -\frac{e}{m}\hat{p}\hat{A} + \frac{e^2}{2m}\hat{A}^2$

- The A^2 term is usually very small except for very intense fields (it corresponds to two-photon processes)
- Then:

$$\boxed{\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(r) - \frac{e}{m}\hat{p}\hat{A} + \sum_k \hbar\omega_k \hat{a}_k^+ \hat{a}_k}$$

* For the moment we have just quantized the electromagnetic field. One can also quantize the electron wave field. This is done in a similar way as that employed for the quantization of the electromagnetic field

As we just saw, the atomic Hamiltonian reads:

$$\hat{H}_A = \frac{\hat{P}^2}{2m} + V(r) = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)$$

The eigenfunctions of $\hat{H}_A$ fulfill

$$\hat{H}_A \psi_j(\vec{r}) = E_j \psi_j(\vec{r}) \qquad \left[\int d^3\vec{r}\, \psi_j^*(\vec{r})\psi_i(\vec{r}) = \delta_{ji}\right]$$

where E_j are the corresponding eigenenergies.

· Any general wavefunction $\psi(\vec{r})$ can be written as a linear combination of eigenfunctions:

$$\psi(\vec{r}) = \sum_j b_j \psi_j(\vec{r})$$

In second quantization we replace the complex amplitudes b_j by operators $\hat{b}_j$:

$$\left.\begin{aligned}\hat{\psi}(\vec{r}) &= \sum_j \hat{b}_j \psi_j(\vec{r}) \\ \hat{\psi}^{+}(\vec{r}) &= \sum_j \hat{b}_j^{+} \psi_j^*(\vec{r})\end{aligned}\right\}\ \underline{\text{Atomic field operators}}$$

· We can then re-write $\hat{H}_A$:

$$\hat{H}_A = \int \hat{\psi}^{+}(\vec{r})\, \hat{H}_A\, \hat{\psi}(\vec{r})\, d^3\vec{r} = \sum_{ij} \hat{b}_i^{+}\hat{b}_j \int \psi_j^*(\vec{r})\left[\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right]\psi_i(\vec{r})\,d^3r$$

$$= \sum_{ij} \hat{b}_i^{+}\hat{b}_j \underbrace{\int d^3r\, \psi_j^*(\vec{r})\psi_i(\vec{r})}_{\delta_{ji}} E_i = \sum_i E_i\, \hat{b}_i^{+}\hat{b}_i$$

* The operators $\hat{b}_j^+$ and $\hat{b}_j$ describe, respectively, the creation and annihilation of one electron in the state j

- let us denote by $|0\rangle$ the vacuum (so no electron)

then $\hat{b}_j^+|0\rangle$ describes one electron in level j.

. Since electrons are fermions, then the Pauli exclusion principle forbids two fermions in the same state (we don't consider spin in this discussion). Hence $\hat{b}_j^+ \hat{b}_j^+ |0\rangle = 0$.

. The ladder operators $\hat{b}_j^+, \hat{b}_j$ obey then fermionic anticommutation rules (contrary to the bosonic commutation rules of e.g. photons):

$$\{\hat{b}_i, \hat{b}_j^+\} = \hat{b}_i \hat{b}_j^+ + \hat{b}_j^+ \hat{b}_i = \delta_{ij}$$

$$\{\hat{b}_i, \hat{b}_j\} = \{\hat{b}_i^+, \hat{b}_j^+\} = 0$$

. For us this will not be a great concern since in our discussions we will just study single-electron states.

* let's re-write now the interaction Hamiltonian using the quantization of the electron wave field:

$$\hat{H}_{INT} = \int d^3r\, \hat{\psi}^+(\vec{r}) \left[-\frac{e}{m} \hat{A}\hat{p}\right] \hat{\psi}(\vec{r})$$

Remember that:

$$\hat{\psi}(\vec{r}) = \sum_j \hat{b}_j \psi_j(\vec{r})$$

$$\hat{\psi}^+(\vec{r}) = \sum_j \hat{b}_j^+ \psi_j^*(\vec{r})$$

$$\hat{A} = \sum_K \sqrt{\frac{\hbar}{2\omega\varepsilon_0}} \left[\hat{a}_K \vec{u}_K(\vec{r}) + \hat{a}_K^+ \vec{u}_K^*(\vec{r})\right]$$

Remember that $\vec{u}_K(\vec{r}) = \underbrace{\frac{e^{i\vec{k}\vec{r}}}{\sqrt{V}}}_{\text{in free space}} \hat{e}$ ← polarization vector

- Then

$$\hat{H}_{INT} = -\frac{e}{m}\sum_{ijk} \hat{b}_i^{\dagger}\hat{b}_j \int d^3r\, \psi_i^*(\vec{r}) \sqrt{\frac{\hbar}{2\omega\epsilon_0}} \left(\hat{a}_k \vec{u}_k(\vec{r}) + \hat{a}_k^{\dagger}\vec{u}_k^*(\vec{r})\right)\cdot\hat{p}\, \psi_j(\vec{r})$$

let

$$g_{ijk} \equiv -\frac{e}{m}\sqrt{\frac{\cancel{1}}{2\omega\epsilon_0\hbar}} \int d^3r\, \psi_i^*(\vec{r}) \left(\vec{u}_k(\vec{r})\cdot\hat{p}\right)\psi_j(\vec{r})$$

Then:

$$\hat{H}_{INT} = \hbar \sum_{ijk} \hat{b}_i^{\dagger}\hat{b}_j \left(g_{ijk}\hat{a}_k + g_{ijk}^*\hat{a}_k^{\dagger}\right)$$

- Then, the <u>full Hamiltonian</u> in second quantization reads

$$\boxed{\hat{H} = \sum_j E_j \hat{b}_j^{\dagger}\hat{b}_j + \sum_k \hbar\omega_k \hat{a}_k^{\dagger}\hat{a}_k + \hbar\sum_{ijk}\hat{b}_i^{\dagger}\hat{b}_j\left[g_{ijk}\hat{a}_k + g_{ijk}^*\hat{a}_k^{\dagger}\right]}$$

- <u>Dipole approximation</u>

- The form of g_{ijk} is relatively complicated and can be easily simplified in typical light sources, with a wavelength much larger than the atomic size (for example the optical part of the spectrum is around 500nm, which must be compared with the Bohr radius $a_{BOHR} \simeq 0.05$ nm, i.e. 10^4 times larger)

- Therefore we can assume that the electron does not see the change in space of the electric field. In other words we can substitute $u_k(\vec{r}) \simeq u_k(\vec{r}_0) \sim e^{i\vec{k}\vec{r}_0}$ where $\vec{r}_0$ are the coordinates of the atom. This is the so-called <u>dipole approximation</u>, and as mentioned above it works very well for typical light sources (it can start failing in the X-ray regime).

• Using this approximation:

$$\int \psi_i^*(\vec r)\,(\vec u_k(\vec r)\,\hat p)\,\psi_j(\vec r)\,d^3r \cong \vec u_k(\vec r_0)\cdot\int \psi_i^*(\vec r)\,\hat p\,\psi_j(\vec r)\,d^3r$$

Since $\hat p = m\frac{d\hat r}{dt} = m\frac{i}{\hbar}[\hat H_A, \hat r]$:

$$\int \psi_i^*(\vec r)\,\hat p\,\psi_j(\vec r)\,d^3r = \frac{im}{\hbar}\int \psi_i^*(\vec r)\,[\hat H_A r - r\hat H_A]\,\psi_j(\vec r)\,d^3r$$

$$= \frac{im}{\hbar}\int d^3r\,\left[E_i\,\psi_i^*(\vec r)\,r\,\psi_j(\vec r) - \psi_i^*(\vec r)\,r\,E_j\,\psi_j(\vec r)\right]$$

$$= \frac{im}{\hbar}(E_i - E_j)\underbrace{\int d^3r\,\psi_i^*(\vec r)\,\vec r\,\psi_j(\vec r)}_{\langle i|\vec r|j\rangle} \longrightarrow \vec d_{ij}$$

$\Rightarrow$ dipole moment for the $j \to i$ transition

$$= \frac{im}{\hbar}(E_i - E_j)\,\vec d_{ij}$$

Then

$$g_{ijk} \cong \frac{-e}{m}\sqrt{\frac{1}{2\hbar\omega\epsilon_0}}\;\vec u_k(\vec r_0)\cdot\frac{im}{\hbar}(E_i - E_j)\,\vec d_{ij}$$

* Note that g_{ijk} gives the coupling strength between a photon of frequency ω and a transition between 2 electronic levels of energies E_i and E_j.

Of energies E_i and E_j depends obviously on ω and $(E_i - E_j)$,

Note that g_{ijk} depends obviously on ω and $(E_i - E_j)$, but it also depends on $\vec u_k(\vec r_0)\cdot\vec d_{ij}$.

Remember that $\vec u_k(\vec r_0) = \frac{e^{i\vec k\vec r_0}}{\sqrt V}\,\hat e$ where $\hat e$ is the polarization vector.

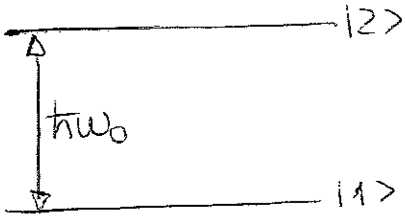
Then the coupling strength depends on $\hat e\cdot\vec d_{ij}$.

Note that due to reasons of symmetry of the wavefunctions it may well be that $\hat e\cdot\vec d_{ij} = \langle i|\hat e\cdot\vec r|j\rangle = 0$.

This introduces important selection rules for allowed transitions for a given polarization.

* Two-level approximation

- We will now further simplify our problem by considering a single-mode of the electromagnetic field (i.e. a photon with some frequency ω, momentum $\hbar\vec{k}$, and polarization $\hat{e}$) interacting with a simplified atom with just two levels $|1\rangle$ and $|2\rangle$



This approximation look quite strong, but it makes full sense in many problems, because e.g. a laser of a given frequency ω is only close to be resonant with some particular transition ω_0.

- Note: an electromagnetic frequency ω is said to be resonant with an electronic transition ω_0, when $\omega = \omega_0$. At resonance the coupling is maximal (as long as $\hat{e}\cdot\vec{d}_{ij} \neq 0$).

- Hence $E_2 - E_1 = \hbar\omega_0$, and the coupling constant for the single mode with the two-level atom is:

$$g \equiv -ie\sqrt{\frac{1}{2\hbar\omega\epsilon_0}}\,\omega_0\,\vec{u}(\vec{r}_0)\cdot\vec{d}_{12} \qquad \left(\begin{array}{l}\text{for simplicity we will choose}\\ u(\vec{r}_0)\text{ such that } g \text{ is real}\end{array}\right)$$

and now the Hamiltonian is significantly simplified:

$$\hat{H}_A = E_1\,\hat{b}_1^+\hat{b}_1 + E_2\,\hat{b}_2^+\hat{b}_2$$

$$\hat{H}_F = \hbar\omega\,\hat{a}^+\hat{a}$$

$$\hat{H}_{INT} = \hbar g\left[\hat{b}_1^+\hat{b}_2\,(\hat{a}+\hat{a}^+) + \hat{b}_2^+\hat{b}_1\,(\hat{a}+\hat{a}^+)\right]$$

* Rotating-wave approximation

* We can further simplify the Hamiltonian by realizing the following.

* let $\hat{H}_0 = \hat{H}_A + \hat{H}_F$

then $\hat{H} = \hat{H}_0 + \hat{H}_{INT}$

* let's work in the interaction picture

$$(\hat{H}_{INT})_I = e^{i\hat{H}_0 t/\hbar}\, \hat{H}_{INT}\, e^{-i\hat{H}_0 t/\hbar}$$

In the interaction picture the operators $\hat{a}$ and $\hat{b}_j$ transform as:

$$\hat{b}_j(t) = \hat{b}_j(0)\, e^{-iE_j t/\hbar}$$
$$\hat{a}(t) = \hat{a}(0)\, e^{-i\omega t/\hbar}$$

then

$$(\hat{H}_{INT})_I = \hbar g \Big\{ \hat{b}_1^+ \hat{b}_2 \hat{a}\, e^{i[E_1 - E_2 - \hbar\omega]t/\hbar} + \hat{b}_1^+ \hat{b}_2 \hat{a}^+ e^{i[E_1 - E_2 + \hbar\omega]t/\hbar} + \hat{b}_2^+ \hat{b}_1 \hat{a}\, e^{i(E_2 - E_1 - \hbar\omega]t/\hbar} + \hat{b}_2^+ \hat{b}_1 \hat{a}^+ e^{i(E_2 - E_1 + \hbar\omega]t/\hbar} \Big\}$$

$$= \hbar g \Big[\hat{b}_1^+ \hat{b}_2 \hat{a}\, e^{-i(\omega_0 + \omega)t} + \hat{b}_1^+ \hat{b}_2 \hat{a}^+ e^{-i(\omega_0 - \omega)t} + \hat{b}_2^+ \hat{b}_1 \hat{a}\, e^{i(\omega_0 - \omega)t} + \hat{b}_2^+ \hat{b}_1 \hat{a}^+ e^{i(\omega_0 + \omega)t} \Big]$$

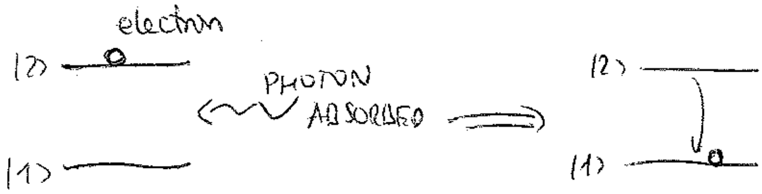
- If $\omega_0 \approx \omega$ which occurs when the electromagnetic field is close to the atomic resonance, the terms with $e^{\pm i(\omega_0 + \omega)t}$ oscillate as $e^{\pm i 2\omega}$, whereas the terms with $e^{\pm i(\omega_0 - \omega)t}$ oscillate much slower. In typical optical transitions,

$$\left.\begin{array}{l} \omega_0 + \omega \sim 10^{14}\ \mathrm{Hz} \\ \omega_0 - \omega \lesssim 10^{6}\ \mathrm{Hz} \end{array}\right\}$$

oscillate much slower. In typical optical transitions, due to this enormous difference of time scales we may neglect those terms oscillating very fast. $\Rightarrow$ This is the so-called ROTATING-WAVE APPROXIMATION (RWA)

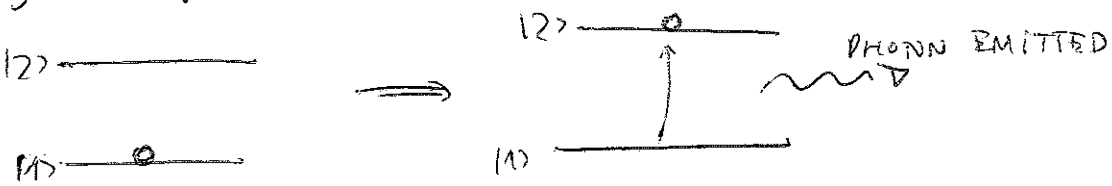
• Another way of looking to the RWA is to realize that those terms which oscillate very fast are those which largely violate energy conservation.

For example: $\hat{b}_1^+ \hat{b}_2 \hat{a} \Rightarrow$ denotes the absorption of a photon and the transition from $|2\rangle$ to $|1\rangle$



Clearly this process violates the energy conservation.

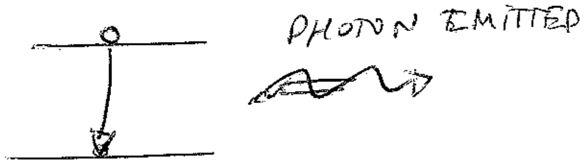
Similarly $\hat{b}_2^+ \hat{b}_1 \hat{a}^+$ denotes a transition from $|1\rangle$ to $|2\rangle$ which give a photon:



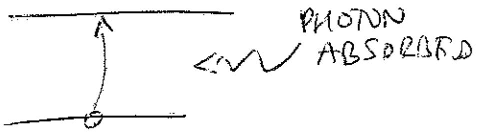
which also violates the energy conservation.

· The terms that survive are those which conserve the energy at most, i.e.

$\hat{b}_1^+ \hat{b}_2 \hat{a}^+ \longrightarrow$



$\hat{b}_2^+ \hat{b}_1 \hat{a} \longrightarrow$



which are those processes that we would intuitively expect.

· After doing the RWA the Hamiltonian becomes

$$\boxed{\hat{H}_{RWA} = E_1 \hat{b}_1^+ \hat{b}_1 + E_2 \hat{b}_2^+ \hat{b}_2 + \hbar\omega \hat{a}^+ \hat{a} + \hbar g \left(\hat{b}_1^+ \hat{b}_2 \hat{a}^+ + \hat{b}_2^+ \hat{b}_1 \hat{a}\right)}$$

• Pseudo-spin. Jaynes-Cummings model

• It's quite convenient to exploit the analogy between a two-level atom and a spin-1/2 particle ($|1\rangle \equiv |\downarrow\rangle$, $|2\rangle \equiv |\uparrow\rangle$)

• Remember that the Pauli spin operators (multiplied by 1/2 for convenience) are

$$\hat{\sigma}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \hat{\sigma}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \hat{\sigma}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can introduce the rising and lowering operators

$$\hat{\sigma}^{\pm} = \hat{\sigma}_x \pm i\,\hat{\sigma}_y$$

which fulfill:

$$[\hat{\sigma}^+, \hat{\sigma}^-] = 2\hat{\sigma}_z$$

$$[\hat{\sigma}^{\pm}, \hat{\sigma}_z] = \mp\hat{\sigma}^{\pm}$$

$$\hat{\sigma}^+\hat{\sigma}^- + \hat{\sigma}^-\hat{\sigma}^+ = \hat{\mathbb{1}}$$

• It's easy to show that

- $\hat{b}_2^+\hat{b}_1$ plays the role of $\hat{\sigma}^+$ (passes from $|\downarrow\rangle$ to $|\uparrow\rangle$)
- $\hat{b}_1^+\hat{b}_2$ plays the role of $\hat{\sigma}^-$ (passes from $|\uparrow\rangle$ to $|\downarrow\rangle$)
- $\frac{1}{2}(\hat{b}_2^+ b_2 - \hat{b}_1^+\hat{b}_1)$ plays the role of $\hat{\sigma}_z$

• Then $\hat{H}_A = \hbar\omega_0\hat{\sigma}_z$ ($E_1 = -\hbar\omega_0/2$, $E_2 = \hbar\omega_0/2$)

$$\hat{H}_{INT} = \hbar g\,(\hat{\sigma}^-\hat{a}^+ + \hat{\sigma}^+\hat{a})$$

Then

$$\boxed{\hat{H} = \hbar\omega_0\hat{\sigma}_z + \hbar\omega\,\hat{a}^+\hat{a} + \hbar g(\hat{\sigma}^-\hat{a}^+ + \hat{\sigma}^+\hat{a})}$$

This is the so-called Jaynes-Cummings Hamiltonian

It's the simplest model for the atom-field interaction.

* This Hamiltonian is very simple and can be readily solved.

let $|n+1,1\rangle \longrightarrow$ state with $n+1$ photons and in the $|1\rangle$ state

$|n,2\rangle \longrightarrow$ state with n photons and in the $|2\rangle$ state

It's easy to see that

$$\hat{H}|n+1,1\rangle = \left[-\frac{\hbar\omega_0}{2} + \hbar\omega(n+1)\right]|n+1,1\rangle + \hbar g\sqrt{n+1}\;|n,2\rangle$$

$$\hat{H}|n,2\rangle = \hbar g\sqrt{n+1}\;|n+1,1\rangle + \left[\frac{\hbar\omega_0}{2} + \hbar\omega n\right]|n,2\rangle$$

i.e. the Hamiltonian does not bring us out of the manifold $\{|n+1,1\rangle, |n,2\rangle\}$. In the basis of these states we can ~~diagonalize~~ write $\hat{H}$ in the matrix form:

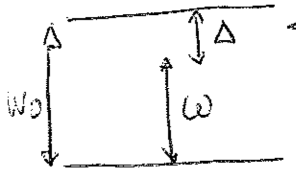
$$\hat{H} = \hbar\omega(n+1/2)\,\hat{\mathbb{1}} + \hbar\begin{bmatrix}-\Delta/2 & g\sqrt{n+1}\\ g\sqrt{n+1} & \Delta/2\end{bmatrix}$$

where $\hat{\mathbb{1}} \equiv$ Identity matrix

$\Delta = \omega_0 - \omega$

This is the so-called <u>detuning</u>

· The detuning Δ tells us how far is the light frequency ω from the atomic transition ω_0:



if $\omega<\omega_0$ (like in the figure) one says that the light ~~system~~ is <u>red-detuned</u>, this is because smaller frequency means larger wavelength (and in the optical spectrum red occupies the largest wavelengths).

If $\omega>\omega_0$ one says that the light is <u>blue detuned</u>.

· One can now diagonalize the matrix above. One has of course two eigenstates

$$|e_1\rangle = \cos\phi\;|n+1,1\rangle + \sin\phi\;|n,2\rangle$$

$$|e_2\rangle = -\sin\phi\;|n+1,1\rangle + \cos\phi\;|n,2\rangle$$

$$\Bigg\}\begin{pmatrix}|e_1\rangle\\|e_2\rangle\end{pmatrix} = \hat{R}(\phi)\begin{pmatrix}|n+1,1\rangle\\|n,2\rangle\end{pmatrix}$$

where $$\tan\phi = \frac{\Delta/2 \pm \sqrt{g^2(n+1)+\Delta^2/4}}{g\sqrt{n+1}}$$

and eigenenergies $\hbar\omega_{1,2}$, where $\omega_{1,2} = \omega(n+\frac{1}{2}) \pm \sqrt{g^2(n+1)+\Delta^2/4}$

* Given an initial state within the manifold $\{|n+1,1\rangle, |n,2\rangle\}$:

$$|\psi(0)\rangle = A_{n+1,1}(0)\,|n+1,1\rangle + A_{n,2}(0)\,|n,2\rangle$$

we can easily obtain $|\psi(t)\rangle$:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \hat{R}(\phi)^{-1}\begin{bmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t}\end{bmatrix}\hat{R}(\phi)\,|\psi(0)\rangle$$

$$= \begin{bmatrix} \cos^2\phi\, e^{-i\omega_1 t} + \sin^2\phi\, e^{-i\omega_2 t} & \cos\phi\sin\phi\,(e^{-i\omega_1 t} - e^{-i\omega_2 t}) \\ \cos\phi\sin\phi\,(e^{-i\omega_1 t} - e^{-i\omega_2 t}) & \cos^2\phi\, e^{-i\omega_2 t} + \sin^2\phi\, e^{-i\omega_1 t}\end{bmatrix}|\psi(0)\rangle$$

* let's consider that we start in the ground state $|1\rangle$, then $A_{n,2}(0) = 0$, $A_{n+1,1}(0) = 1$. The probability to find the atom in the state $|2\rangle$ is given by:

$$|\langle n,2|\psi(t)\rangle|^2 = \cos^2\phi\sin^2\phi\,|e^{-i\omega_1 t} - e^{-i\omega_2 t}|$$

$$= (2\sin\phi\cos\phi)^2 \sin^2\left[\left(\frac{\omega_1-\omega_2}{2}\right)t\right]$$

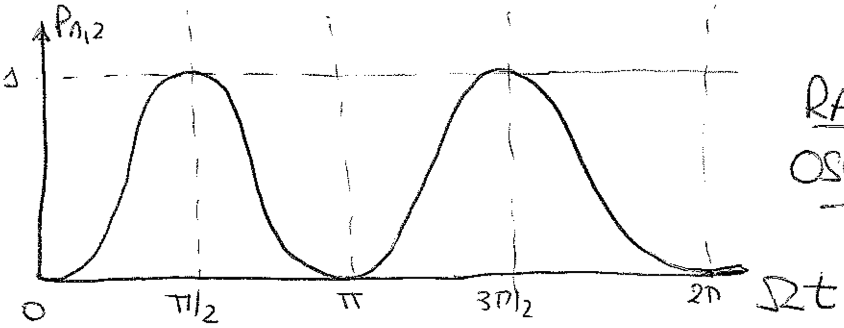
$$= \left[\frac{2\tan\phi}{1+\tan^2\phi}\right]^2 \sin^2\left[t\sqrt{g^2(n+1)+\Delta^2/4}\right]$$

Note that the population in $|2\rangle$ performs an oscillation between 0 and $\left(\frac{2\tan\phi}{1+\tan^2\phi}\right)^2$ with a frequency $\sqrt{g^2(n+1)+\Delta^2/4} = \Omega_G$.

Ω_G is the so-called <u>generalized Rabi frequency</u>.

If $\Delta = 0$ (i.e. if $\omega = \omega_0$, i.e. if the light is <u>in resonance</u> with the transition), then $|\langle n,2|\psi(t)\rangle|^2 = \sin^2[g\sqrt{n+1}\,t]$, i.e. the population of $|2\rangle$ performs an oscillation from 0 to 1 with a frequency $\Omega = g\sqrt{n+1}$ $\equiv$ <u>Rabi frequency</u>

* So, graphically $P_{n,2}(t) = |\langle n,2|\psi(t)\rangle|^2$



RABI OSCILLATIONS

· Up to now we have just considered the set $\{|n+1,1\rangle, |n,2\rangle\}$. From this knowledge we can easily calculate the behavior of the two-level atom when interacting with a coherent-state of light. Assuming, as before, that the atom is in the ground state initially, then the probability to be after t in $|2\rangle$ is:

$$P_2(t) = \sum_{n=0}^{\infty} P_{\bar{n}}(n)\, P_{n,2}(t)$$

where $P_{\bar{n}}(n) = \dfrac{e^{-\bar{n}}\,\bar{n}^n}{n!}$ is a Poissonian distribution with mean number $\bar{n}$ (remember that in a poissonian distribution the variance $\Delta n = \sqrt{\bar{n}}$). For a coher. state $|\alpha\rangle \to \bar{n} = |\alpha|^2$

· Then (for $\Delta=0$) $P_2(t) = \sum_{n=0}^{\infty} P_{\bar{n}}(n) \sin^2 g\sqrt{n+1}\, t = \frac{1}{2}\sum_{n=0}^{\infty} P_{\bar{n}}(n)\left[-\cos 2g\sqrt{n+1}\,t + 1\right]$

$$= \frac{1}{2}\left[1 - \sum_{n=0}^{\infty} P_{\bar{n}}(n)\cos 2g\sqrt{n+1}\, t\right]$$

Due to the Poissonian distribution of the photon number there is a spread in the Rabi frequencies ($\Delta n \sim \bar{n}^{1/2}$). As a result the Rabi oscillations will collapse after some oscillations due to the destructive interference between the different frequencies.

· Let's see this more clearly. Let's consider times $gt \ll \sqrt{\bar{n}}$, and let's assume $\bar{n} \gg \sqrt{\bar{n}}$

then $\cos[2g\sqrt{n+1}\,t] = \cos\left[2g\sqrt{\bar n+1}\left[1+\frac{(n-\bar n)}{\bar n+1}\right]^{1/2} t\right] =$

$$\simeq \cos\left[2g\sqrt{\bar n+1}\left(1+\frac{(n-\bar n)}{2(\bar n+1)}\right)t\right]$$

$$\simeq \cos[2g\sqrt{\bar n+1}\,t]\cos\left[g\frac{(n-\bar n)}{\sqrt{\bar n+1}}t\right] - \sin(2g\sqrt{\bar n+1}\,t)\sin\left(\frac{g(n-\bar n)t}{\sqrt{\bar n+1}}\right)$$

$$\simeq \cos(2g\sqrt{\bar n+1}\,t)\left[1-\frac{1}{2}\frac{g^2(n-\bar n)^2t^2}{\bar n+1}\right] - \sin[2g\sqrt{\bar n+1}\,t]\,\frac{g(n-\bar n)t}{\sqrt{\bar n+1}}$$

- For $\bar n \gg 1 \to P_{\bar n}(n) \simeq \dfrac{e^{-(n-\bar n)^2/2\bar n}}{\sqrt{2\bar n\pi}}$ as we already saw.

Then:

$$\sum_{\bar n} P_{\bar n}(n)(n-\bar n)^2 \simeq \int_{-\infty}^{\infty} dn\, \frac{e^{-(n-\bar n)^2/2\bar n}}{\sqrt{2\bar n\pi}}(n-\bar n)^2 = \bar n$$

$$\sum_{\bar n} P_{\bar n}(n)(n-\bar n) \simeq 0$$

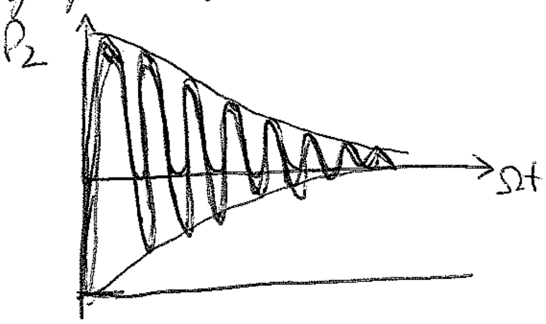
Then:

$$P_2(t) \cong \frac{1}{2}\left\{1-\cos[2g\sqrt{\bar n+1}\,t]\left(1-\frac{1}{2}\frac{g^2\bar n\,t^2}{\bar n+1}\right)\right\}$$

$$\simeq \frac{1}{2}\left[1-\cos[2g\sqrt{\bar n+1}\,t]\,e^{-g^2\bar n\,t^2/2(\bar n+1)}\right]$$

$$\simeq \frac{1}{2}\left[1-\cos[2g\sqrt{\bar n+1}\,t]\,e^{-g^2t^2/2}\right]$$

Graphically:



The Rabi oscillations occur under a gaussian envelope. The characteristic collapse time is then

$$t_{collapse} \sim 1/g$$

* The number of oscillations before the collapse occurs is $\sim\sqrt{\bar{n}}$

* A more accurate evaluation shows a partial <u>revival</u> of the initial oscillations after a time

$$t_{revival} \sim \frac{2\pi}{g}\sqrt{\bar{n}}$$

Thus a quasi-periodic burst of Rabi oscillations occurs after approximately $\bar{n}$ Rabi periods.

The existence of periodic revivals arises from the discreteness of the sum over n, which ensures that after some finite time all oscillating terms come more or less back in phase with each other. Since the frequencies $g\sqrt{n+1}$ are irrational and thus incommensurate the rephasing is not perfect, and the revival is just partial.

The revival is hence a pure quantum effect.

• Note that the collapse occurs after $\sim\sqrt{\bar{n}}$ oscillations. If $\bar{n}$ is huge, this means that the collapse will occur only after a very large number of oscillations. The average number of photons in a e.g. laser can be really huge, and hence in typical time scales one observes perfect Rabi oscillations without damping (since $\bar{n} >>>> \sqrt{\bar{n}}$), and with frequency

$$\Omega = g\sqrt{\bar{n}}.$$

* In our analysis of the light-atom interaction we have not considered the spontaneous decay of the upper state into the lower one. We will postpone this analysis for later in the course, once we have introduce the master equations.

* If $\bar{n} \gg \sqrt{\bar{n}}$ we can actually forget the quantum nature of the EM field

$$\Omega \simeq g\sqrt{\bar{n}} \overset{\omega_0 \approx \omega}{\simeq} \frac{\vec{E}\cdot(e\vec{d}_{12})}{\hbar}$$

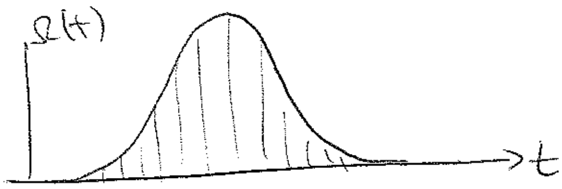
The Rabi frequency is hence given by the coupling of the electric field and the dipole of the 2-level transition.

. The Hamiltonian for the 2-level atom becomes

$$\hat{H} = \hbar \begin{pmatrix} -\Delta/2 & \Omega \\ \Omega & \Delta/2 \end{pmatrix}$$

. Let's consider the case in resonance $\hat{H} = \hbar\Omega\hat{\sigma}_x$.

A curious property of this Hamiltonian can be seen when $\Omega = \Omega(t)$:



This is the case of a pulse of laser light

Then $\hat{H} = \hbar\Omega(t)\hat{\sigma}_x$; let $|\psi(t)\rangle = \psi_1(t)|1\rangle + \psi_2(t)|2\rangle$

Hence $\frac{d}{dt}\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} = -i\Omega(t)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix}$; let $\underbrace{\bar{\Omega} = \int \Omega(t)\,dt}_{\text{Area of the pulse}}$

Then $\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} = e^{-i\bar{\Omega}\hat{\sigma}_x}\begin{pmatrix}\psi_1(0)\\ \psi_2(0)\end{pmatrix}$

$e^{-i\bar{\Omega}\hat{\sigma}_x} = \cos\bar{\Omega} - i\sin\bar{\Omega}\,\hat{\sigma}_x$; let $\psi_1(0) = 1$; $\psi_2(0) = 0$

Hence $\psi_2(t) = -i\sin\bar{\Omega} \longrightarrow \boxed{|\psi_2(t)|^2 = \sin^2\int\Omega(t)\,dt}$

The population in $|2\rangle$ just depends on the area of the pulse!

E.g. for $\bar{\Omega} = \pi/2 \rightarrow |\psi_2(t)|^2 = 1$; This is the so-called <u>AREA LAW</u>