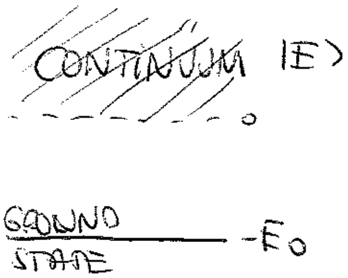
• COHERENCE PROPERTIES OF THE ELECTROMAGNETIC FIELD

· In this new block of lectures we will study the concept of optical coherence. For this we will need to introduce the idea of correlations. But first of all we will try to make a toy-model of photon detection.

· We consider a one electron atom with ground state $-E_0$ and states in the continuum with energies $E \geq 0$.



An incident electromagnetic field couples $|0\rangle$ to the continuum causing ionization. The photocurrent due to the ionization is then detected (probably after passing through a multiplier, but we don't care here about that).

We can proceed in a similar way as we just did when discussing the light-atom interaction. The only difference is that now we deal with a continuum of states, and hence we will transform sums into integrals.

In the interaction picture with respect to the EM field

$$\hat{H} = \underbrace{-E_0\,|0\rangle\langle 0| + \int_0^\infty dE\,E\,|E\rangle\langle E|}_{\hat{H}_A} + \hbar k \int_0^\infty dE\,|0\rangle\langle E|\,\mathcal{E}^{(-)}(\vec{r},t) + \hbar k \int_0^\infty dE\,|E\rangle\langle 0|\,\mathcal{E}^{(+)}(\vec{r},t)$$

EMISSION

ABSORPTION

Here we have employed already the dipole approximation and the rotating wave approximation. We are working in the interaction picture with respect to the EM field (this is why there's no $\hat{H}_F$ now), and hence $\mathcal{E}^{-}, \mathcal{E}^{+}$ are function of t ($\sim e^{i\omega t}, e^{-i\omega t}$).

$\mathcal{E}^{-}(\vec{r},t)$ is ~~the~~ part of the electric field going with $e^{i\omega_L t}$
(goes with creation operators of photons when second-quantized)
$\mathcal{E}^{+}(\vec{r},t)$ goes with $e^{-i\omega_L t}$ (and destruction operators when 2nd-quantized)

For simplicity we assume a single coupling constant κ.

* The electron state is given by

$$|\psi(t)\rangle = \alpha(t)\,|0\rangle + \int_0^\infty dE\;\beta(E,t)\,|E\rangle$$

$$\hat{H}\,|\psi(t)\rangle = i\hbar\,\frac{\partial}{\partial t}|\psi(t)\rangle$$

Hence:

$$\dot{\alpha} = -i\kappa\int dE\;\beta(E,t)\,\mathcal{E}^{-}(\vec{r},t)$$

$$\dot{\beta}_{(E,t)} = -\frac{i}{\hbar}(E+E_0)\,\beta(E,t) - i\kappa\,\mathcal{E}^{+}(\vec{r},t)\,\alpha(t)$$

(Note: for simplicity of the notation we choose the ground state energy as the origin of energies. Then $E \longrightarrow E+E_0$)

Let $\beta = \tilde{\beta}\,e^{-i/\hbar\,(E+E_0)t}$ (this is equivalent to passing into interaction picture with H_A)

Then:

$$\dot{\alpha} = -i\kappa\int dE\;\tilde{\beta}(E,t)\,e^{-i/\hbar\,(E+E_0)t}\,\mathcal{E}^{-}(\vec{r},t)$$

$$\tilde{\beta}(E,t) = -i\kappa\int dt'\;\mathcal{E}^{+}(\vec{r},t')\,\alpha(t')\,e^{-i/\hbar\,(E+E_0)t'}$$

Hence:

$$\dot{\alpha} = -\kappa^2\int dE\;e^{-i/\hbar\,(E+E_0)t}\,\mathcal{E}^{-}(\vec{r},t)\int dt'\;\mathcal{E}^{+}(\vec{r},t')\,\alpha(t')\,e^{-i/\hbar\,(E+E_0)t'}$$

$$= -\kappa^2\int_0^t dt'\;\mathcal{E}^{-}(\vec{r},t)\,\mathcal{E}^{+}(\vec{r},t')\left[\int_0^\infty dE\;e^{-i/\hbar\,(E+E_0)(t-t')}\right]\alpha(t')$$

Since $\mathcal{E}^{\pm} \sim e^{\mp i\omega_L t}$, then only energies $E \sim E_0+\hbar\omega_L$ will significantly contribute. Hence we can safely approximate

$$\int_0^\infty dE\;e^{-i/\hbar\,E(t-t')} \approx \int_{-\infty}^{\infty} dE\;e^{-i/\hbar\,E(t-t')} = 2\pi\hbar\,\delta(t-t')$$

* Then:

$$\dot{\alpha}(t) = -k^2 \int_0^t dt' \, \mathcal{E}^-(\vec{r},t)\, \mathcal{E}^+(\vec{r},t')\, 2\pi\hbar\, \delta(t-t')\, \alpha(t') = \qquad \int_0^t dt' \delta(t-t') f(t') = \frac{f(t)}{2}$$

$$= -\pi\hbar k^2 \, \mathcal{E}^-(\vec{r},t)\, \mathcal{E}^+(\vec{r},t)\, \alpha(t) = -\Gamma(t)\, \alpha(t)$$

Hence $\Gamma(t)$ is the instantaneous ionization rate, i.e. it tells us how the ground-state is transferred into the continuum, and hence how it's ionized.

· Note that the ionization rate is proportional to the intensity

$$I(\vec{r},t) = \mathcal{E}^-(\vec{r},t)\, \mathcal{E}^+(\vec{r},t)$$

· When the fields are quantized, as we did already, the intensity and hence the detector signal is given by the average

$$\langle \hat{\mathcal{E}}^-(\vec{r},t)\, \hat{\mathcal{E}}^{(+)}(\vec{r},t) \rangle$$

Remember that $\hat{\mathcal{E}}^- \propto \hat{a}^\dagger$, $\hat{\mathcal{E}}^+ \propto \hat{a}$.

The detection is then sensitive to $\sim \hat{a}^+ \hat{a}$. This is an example of normal ordering (i.e. creation operators to the left of destruction operators).

Normal ordering is actually the consequence of energy conservation (i.e. of the rotating-wave approximation).

· The previous expected value is an example of normally ordered correlation function, which is what photodetectors measure.

· In the following we will get deeper into the idea of correlation functions.

• CORRELATION FUNCTIONS

• Correlation functions play a key role in physics. Typically if we deal with some physical fluctuating field $\varphi(\vec{x},t)$ we will be interested in e.g. $\langle \varphi(x,t) \rangle$, but also in $\langle \varphi(\vec{x}_1,t_1)\varphi(\vec{x}_2,t_2)\rangle$. When we deal with quantum field one has to be careful with the order since $[\hat{\varphi}(\vec{x},t_1),\hat{\varphi}(\vec{x}_2,t_2)] \neq 0$ in general. These appropriately ordered quantities are then called correlation functions.

• As commented before we will be interested here in correlation functions involving normally ordered operators formed by combination of $\hat{\mathcal{E}}^+$ and $\hat{\mathcal{E}}^-$.

• We define two-point correlation functions as:

$$G^{(1)}(x;x') \equiv \langle \hat{\mathcal{E}}^-(x)\,\hat{\mathcal{E}}^+(x')\rangle$$

we use the simplified notation $x \equiv (\vec{x},t)$

Note that the signal of the photodetector discussed before was hence proportional to $G^{(1)}(x;x)$.

$G^{(1)}$ is the so-called 1^{st} order correlation function of the radiation field, and (as we will see in a moment) it is sufficient to account for classical interference experiments.

• The description of experiments involving intensity correlation (as the Hanbury-Brown and Twiss experiment discussed later) demands the definition of higher-order correlation functions. We define the n-th correlation function of the EM field as:

$$G^{(n)}(x_1,x_n;x_{n+1}\dots x_{2n}) = \langle \hat{\mathcal{E}}^-(x_1)\dots\hat{\mathcal{E}}^-(x_n)\,\hat{\mathcal{E}}^+(x_{n+1})\dots\hat{\mathcal{E}}^+(x_{2n})\rangle$$

* The correlation functions fulfilled some interesting properties, which can be derived from the property

$\langle \hat{A}^{+}\hat{A} \rangle \geq 0$ where $\hat{A}$ is an operator

which follows from the non-negative character of $\hat{A}^{+}\hat{A}$

(Note: If we are considering a pure quantum state $\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle$. However if we are dealing with a mixed state characterized by a density matrix $\hat{\rho} = \sum_i p_i |i\rangle\langle i|$, then $\langle \hat{O} \rangle \equiv Tr\{\hat{\rho}\hat{O}\}$. Remember e.g. how do you define averages in quantum statistical mechanics)

- Let $\hat{A} = \hat{\mathcal{E}}^{+}(x)$, then $G^{(1)}(x,x) \geq 0$

In general if $\hat{A} = \hat{\mathcal{E}}^{(+)}(x_n) \dots \hat{\mathcal{E}}^{(+)}(x_1)$ then

$$G^{(n)}(x_1 \dots x_n; x_n \dots x_1) \geq 0$$

* Let's choose now

$$\hat{A} = \sum_{j=1}^{n} \lambda_j \, \hat{\mathcal{E}}^{(+)}(x_j) \qquad \lambda_j \equiv \text{complex numbers}$$

Then

$$\langle \hat{A}^{+}\hat{A} \rangle = \sum_{i,j=1}^{n} \lambda_i^{*} \lambda_j \, G^{(1)}(x_i, x_j) \geq 0$$

This means that $G^{(1)}(x_i, x_j)$ regarded as a matrix must be positively defined. Such matrices fulfill that they have a positive determinant:

$$\left| G^{(1)}(x_i, x_j) \right| \geq 0$$

- Let's consider the case $n=1$, then $G^{(1)}(x_1, x_1) \geq 0$ as we already knew.

- Let $n=2$: $\begin{vmatrix} G^{(1)}(x_1 x_1) & G^{(1)}(x_1 x_2) \\ G^{(1)}(x_2 x_1) & G^{(1)}(x_2 x_2) \end{vmatrix} = G^{(1)}(x_1 x_1)\, G^{(1)}(x_2 x_2) - G^{(1)}(x_1 x_2)^* G^{(1)}(x_1 x_2)$

(we use $G^{(1)}(x_1 x_2)^* = G^{(1)}(x_2, x_1)$)

Then $\boxed{G^{(1)}(x_1,x_1)\, G^{(1)}(x_2,x_2) \geq |G^{(1)}(x_1,x_2)|^2}$

We will comment on the physical meaning of this inequality in a short while.

- Using similar procedure, but now with

$$\hat{A} = \lambda_1 \hat{\mathcal{E}}^+(x_1) \dots \hat{\mathcal{E}}^+(x_n) + \lambda_2 \hat{\mathcal{E}}^+(x_{n+1}) \dots \hat{\mathcal{E}}^+(x_{2n})$$

one gets to:

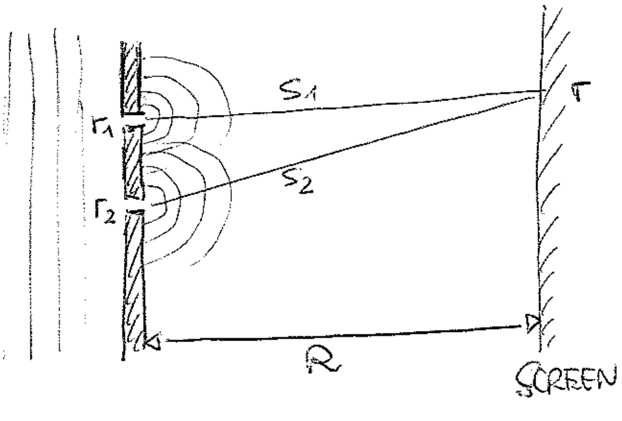
$$\boxed{G^{(n)}(x_1 .. x_n; x_n .. x_1)\, G^{(n)}(x_{n+1} .. x_{2n}; x_{2n} \dots x_{n+1}) \geq |G^{(n)}(x_1 .. x_n; x_{n+1} \dots x_{2n})|^2}$$

which generalizes the expression for 2 points.

- In the following we will employ correlation functions to describe experiments and concepts of crucial importance for the idea of optical coherence.

• YOUNG'S DOUBLE-SLIT EXPERIMENT

- Let's consider first the famous Young's interference experiment.



- A plane wave hits a screen with 2 pinholes. The pin holes act as a source of spherical waves.
- The light is detected at a screen (we assume $s_1, s_2 \gg |r_2 - r_1|$)

$$s_j = |\vec{r}_j - \vec{r}|$$

. The field incident on the screen at position $\vec{r}$ and time t is the superposition of the field created at the two pin holes

$$\mathcal{E}^{+}(\vec{r},t) = \mathcal{E}_1(\vec{r},t) + \mathcal{E}_2(\vec{r},t)$$

where $\mathcal{E}_j^{+}(\vec{r},t)$ is the field produced by the j-th pin hole at the screen

$$\mathcal{E}_j^{+}(\vec{r},t) = \underbrace{\mathcal{E}_j^{+}(\vec{r}_j, t-\tfrac{S_j}{c})}_{\text{field at the } j^{th} \text{ pin hole}} \underbrace{\frac{1}{S_j} e^{ikS_i}}_{\text{spherical wave}} e^{-i\omega \underbrace{\scriptstyle\frac{S_i}{c}}} \longrightarrow \text{time} = \frac{S_i}{c}$$

from slit to screen

* Since $K = \omega/c$, and $S_j \approx R$ (since $R \gg |r_2 - r_1|$), then

$$\mathcal{E}^{+}(\vec{r},t) \cong \frac{1}{R}\left[\mathcal{E}_1^{+}(x_1) + \mathcal{E}_2^{+}(x_2)\right]$$

where $x_j \equiv (\vec{r}_1, t - S_j/c)$

. The intensity observed on the screen is proportional to

$$I = \langle \mathcal{E}^{-}(\vec{r},t)\,\mathcal{E}^{+}(\vec{r},t)\rangle \propto \langle(\mathcal{E}_1^{-}(x_1) + \mathcal{E}_2^{-}(x_2))(\mathcal{E}_1^{+}(x_1) + \mathcal{E}_2^{(+)}(x_2))\rangle$$

$$= \underbrace{G^{(1)}(x_1,x_1) + G^{(1)}(x_2,x_2)}_{\text{intensities of each pinhole in the absence of the other}} + \underbrace{2\,\mathrm{Re}\{G^{(1)}(x_1,x_2)\}}_{\text{Interference term}}$$

Writing $G^{(1)}(x_1,x_2) = |G^{(1)}(x_1,x_2)|\, e^{i\phi(x_1,x_2)}$

$$I = G^{(1)}(x_1,x_1) + G^{(1)}(x_2,x_2) + 2|G^{(1)}(x_1,x_2)|\cos\phi(x_1,x_2)$$

The interference fringes come from the cosine term.
The envelope of the fringes is given by $|G^{(1)}(x_1,x_2)|$

* Note that now we can associate a clear physical meaning to the inequality described previously:

$$|G^{(1)}(x_1 x_2)|^2 \leq G^{(1)}(x_1 x_1)\, G^{(1)}(x_2 x_2)$$

Then, in the interference, the maxima fulfill:

$$I_{max} = G^{(1)}(x_1 x_1) + G^{(1)}(x_2 x_2) + 2|G^{(1)}(x_1 x_2)|$$

$$\leq G^{(1)}(x_1 x_1) + G^{(1)}(x_2 x_2) + 2\left[G^{(1)}(x_1 x_1) G^{(1)}(x_2 x_2)\right]^{1/2}$$

$$\leq \left[G^{(1)}(x_1 x_1)^{1/2} + G^{(1)}(x_2 x_2)^{1/2}\right]^2$$

The interference term cannot be larger than the geometric mean of the intensities $G^{(1)}(x_j, x_j)$.

. The interference experiment leads us directly to the idea of coherence.

• COHERENCE

. The idea of coherence in optics was first associated to the possibility of producing interferences when two fields are superposed. The degree of coherence was then associated with the fringe visibility. From the expression above it becomes clear that the larger $|G^{(1)}(x_1, x_2)|$ the larger the visibility.

. Let's introduce the normalized correlation function:

$$g^{(1)}(x_1, x_2) = \frac{G^{(1)}(x_1, x_2)}{\left[G^{(1)}(x_1, x_1)\, G^{(1)}(x_2, x_2)\right]^{1/2}}$$

From the inequalities developed before $0 \leq |g^{(1)}(x_1, x_2)| \leq 1$

$|g^{(1)}(x_1 x_2)| = 0 \longrightarrow$ the fields are incoherent

$|g^{(1)}(x_1 x_2)| = 1 \longrightarrow$ the fields are maximally coherent.

* The visibility of the fringes is defined as

$$v = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \qquad \left(I_{\substack{max \\ min}} \equiv \substack{\text{maximal} \\ \text{minimal}} \text{ intensity of the fringe pattern} \right)$$

It's clear to see that in the experiment described before $v = |g^{(1)}|$.

· A more general definition of first-order coherence of a field $E(x)$ is that

$$G^{(1)}(x_1; x_2) = \langle \mathcal{E}^-(x_1)\, \mathcal{E}^+(x_2) \rangle = \langle \mathcal{E}^-(x_1) \rangle \langle \mathcal{E}^+(x_2) \rangle$$

i.e. we can factorize $G^{(1)}$. It's easy to see that this immediately leads to $g^{(1)} = 1$.

· We may generalize to give the condition for n^{th} optical coherence:

$$G^{(n)}(x_1 \ldots x_m; x_{n+1} \ldots x_{2n}) = \langle \mathcal{E}^-(x_1) \rangle \cdots \langle \mathcal{E}^-(x_n) \rangle \langle \mathcal{E}^+(x_{n+1}) \rangle \cdots \langle \mathcal{E}^+(x_{2n}) \rangle$$

· Let's write now the intensity pattern at the screen as a function of the creation and annihilation operators of the modes coming from each pin hole.

· Remember that

$$\vec{\mathcal{E}} = -i \left(\frac{\hbar\omega}{2\epsilon_0} \right)^{1/2} \left[\hat{a}^+ e^{i\omega t} \vec{u}_k^*(\vec{r}) - \hat{a}\, e^{-i\omega t} \vec{u}_k(\vec{r}) \right]$$

The pinholes act as sources for single modes of spherical radiation (Huygen's principle). Then

$$\vec{u}_k(\vec{r}) = \frac{1}{\sqrt{4\pi L}} \frac{e^{ikr}}{r} \hat{e} \qquad \left(\begin{array}{l} L \equiv \text{radius of the normalization volume} \\ \hat{e} \equiv \text{polarization vector} \\ e^{ikr}/r \equiv \text{spherical wave} \end{array} \right)$$

Then the field on the screen at position $\vec{r}$ and time t coming from the pin hole i is:

$$\vec{\mathcal{E}}_i^{(+)}(\vec{r}, t) = \hat{e}\, i \left(\frac{\hbar\omega}{2\epsilon_0} \right)^{1/2} \frac{e^{-i\omega t}}{\sqrt{4\pi L}} \frac{e^{ik(\vec{r} - \vec{r}_i)}}{|\vec{r} - \vec{r}_i|} \hat{a}_i \approx \hat{e}\, i \left(\frac{\hbar\omega}{2\epsilon_0} \right)^{1/2} \frac{e^{-i\omega t}}{\sqrt{4\pi L} \cdot R} e^{ikS_i} \hat{a}_i$$

· let $f(\vec{r},t) = i\left(\frac{\hbar\omega}{2\epsilon_0}\right)^{1/2} \frac{e^{-i\omega t}}{\sqrt{4\pi L}\; R}$

Then $\vec{\mathcal{E}}_i^{(+)}(\vec{r},t) = \hat{e}\, f(\vec{r},t)\, e^{ikS_i}\, \hat{a}_i$

· The total field is the sum of the fields coming from each pin hole.

(Note: we will assume that the fields coming from both pin holes have the same polarization vector $\hat{e}$. What if they don't? ⇒ EXERCISE)

· $\mathcal{E}^{(+)}(\vec{r},t) = \mathcal{E}_1^{(+)}(\vec{r},t) + \mathcal{E}_2^{(+)}(\vec{r},t)$

$= f(\vec{r},t)\left[e^{ikS_1}\hat{a}_1 + e^{ikS_2}\hat{a}_2\right]$

(we forget the vector character because both modes have the same polarization)

· Then

$$I = \langle \mathcal{E}^{(-)}(\vec{r},t)\,\mathcal{E}^{(+)}(\vec{r},t)\rangle = |f(\vec{r},t)|^2 \Big\{ \langle \hat{a}_1^\dagger \hat{a}_1\rangle + \langle \hat{a}_2^\dagger \hat{a}_2\rangle + e^{-ik(S_1-S_2)}\langle \hat{a}_1^\dagger \hat{a}_2\rangle + e^{+ik(S_1-S_2)}\langle \hat{a}_1^\dagger \hat{a}_2\rangle^* \Big\}$$

$$= |f(\vec{r},t)|^2 \Big\{ \langle \hat{a}_1^\dagger \hat{a}_1\rangle + \langle \hat{a}_2^\dagger \hat{a}_2\rangle + 2\,|\langle \hat{a}_1^\dagger \hat{a}_2\rangle| \cos\Phi \Big\}$$

where $\Phi \equiv k(S_1-S_2)+\phi$; $\langle \hat{a}_1^\dagger \hat{a}_2\rangle = |\langle \hat{a}_1^\dagger \hat{a}_2\rangle|\, e^{-i\phi}$

· Once more we recover the typical interference fringes with maxima when $k(S_1-S_2)+\phi = 2n\pi$.

· Let's consider a special (and very interesting) case, namely the case in which <u>only one photon</u> impinges on the pinholes:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right)$$

$|n_1,n_2\rangle$ means $\begin{cases} n_1 \text{ photons in mode } 1 \\ n_2 \text{ photons in mode } 2 \end{cases}$

This state denotes that we don't know which pin hole the photon goes through.

· Then $\langle \hat{a}_1^+ \hat{a}_1 \rangle = \frac{1}{2} = \langle \hat{a}_2^+ \hat{a}_2 \rangle$

$$\langle \hat{a}_1^+ \hat{a}_2 \rangle = \frac{1}{2}$$

Hence

$$I = |f(\bar{r},t)|^2 \, (1 + \cos \Phi)$$

Amazingly, it becomes clear from this equation that we can build an interference pattern from a sucession of single-photon interference experiments. <u>The photon interferes with itself</u> !!!

* Another interesting example is a coherent state

$|\psi\rangle = |\alpha_1, \alpha_2\rangle$ with $\alpha_j = |\alpha_j| e^{i\varphi_j}$

Then $\langle \hat{a}_1^+ \hat{a}_1 \rangle = |\alpha_1|^2$

$$\langle \hat{a}_2^+ \hat{a}_2 \rangle = |\alpha_2|^2$$

$$\langle \hat{a}_1^+ \hat{a}_2 \rangle = \alpha_1^* \alpha_2 = |\alpha_1||\alpha_2| \, e^{-i(\phi_1 - \phi_2)}$$

Then:

$$I(\bar{r},t) = |f(\bar{r},t)|^2 \left[|\alpha_1|^2 + |\alpha_2|^2 + 2|\alpha_1||\alpha_2| \cos\left[k(s_1 - s_2) + (\phi_1 - \phi_2) \right] \right]$$

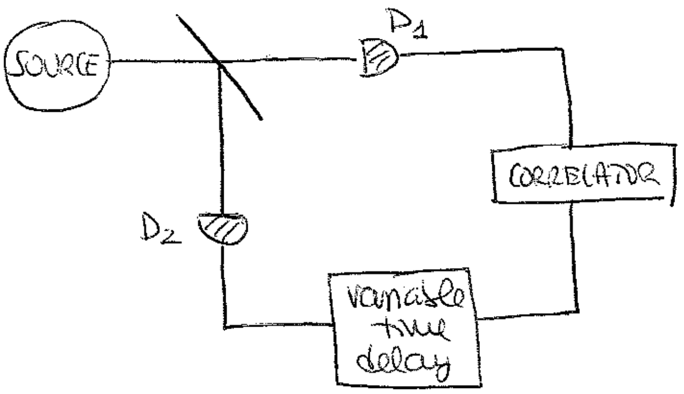
This example shows that one can obtain interferences between independent light beams (i.e. one doesn't need that the beams are coming from the light coming when one and the same beam impinges the pin holes). However, a requirement to observe the interference fringes is that the relative phase between the beams should be locked (i.e. constant) or very slowly varying. Otherwise it becomes clear from the expression above, that if $\phi_1 - \phi_2$ variates quickly the interference pattern is destroyed (i.e. the fields became incoherent).

For example, for a Fock state $|n_1, n_2\rangle$ the relative phase is totally randomized, and not surprisingly (check it!) the interference pattern disappears.

- PHOTON CORRELATION MEASUREMENTS

* Young's interference fringes can be explained by the interference of classical waves, so in this sense first-order correlation measurements do not distinguish between quantum and classical theories of light.

* We turn now to the Hanbury-Brown and Twiss experiment which probes higher-order coherence properties of the field. In this experiment a beam of light (from a star in the original experiment) is split into two beams, which are detected by detectors D_1 and D_2. The signals are then multiplied and averaged in a correlator.



. This procedure differs from the Young 2-slit experiment in that light intensities, rather than amplitudes, are compared. Two absorption measurements are performed on the same field, one at time t and the other with some delay at time $t+\tau$.

. The measured quantity is actually the normally ordered correlation function:

$$G^{(2)}(t, t+\tau, t+\tau, t) = \langle E^{(-)}(t)\, E^{(-)}(t+\tau)\, E^{(+)}(t+\tau)\, E^{(+)}(t) \rangle$$

(we forget the position $\vec{r}$, because now the detection is done at fixed points)

what we are measuring is the coincidence that one detector clicks at time t and the other at time $t+\tau$.

* let's consider first the case of a stationary field. In a stationary field $G^{(1)}(t,t) = G^{(1)}(t',t') = G^{(1)}(0,0)$

$$G^{(2)}(t, t+\tau, t+\tau, t) = G^{(2)}(0, \tau, \tau, 0) \equiv G^{(2)}(\tau)$$

i.e. it's not important the absolute time but the time difference.

· **Note**: Note that for stationary fields

$$G^{(1)}(t_1,t_2) = G^{(1)}(t_1-t_2)$$

Since first-order coherence implies $G^{(1)}(t_1-t_2) = \langle \mathcal{E}^{(-)}(t_1)\rangle\langle \mathcal{E}^{(+)}(t_2)\rangle$

This means that $\mathcal{E}^{-}(t) \sim e^{+i\omega t}$, i.e. the light must have just one frequency to be 1st order coherent. I.e. for a stationary field, 1st order coherence demands a monochromatic source.

(note that a monochromatic source is always 1st order coherent (and in general n^{th} order coherent), but the viceversa is just true for stationary fields)

---

* So, for a stationary field

$$G^{(2)}(\tau) = \langle \mathcal{E}^{-}(0)\,\mathcal{E}^{-}(\tau)\,\mathcal{E}^{+}(\tau)\,\mathcal{E}^{+}(0)\rangle$$

It's convenient to normalize the 2nd order correlation function as:

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{|G^{(1)}(0)|^2}$$

Second-order coherence implies $G^{(2)}(\tau) = \langle \mathcal{E}^{-}(0)\rangle\langle \mathcal{E}^{-}(\tau)\rangle\langle \mathcal{E}^{+}(\tau)\rangle\langle \mathcal{E}^{+}(0)\rangle$

$$= (G^{(1)}(0))^{*}\,(G^{(1)}(\tau))^{*} = \underbrace{G^{1}(0)^2}_{\text{stationary field}}$$

Then $\boxed{g^{(2)}(\tau) = 1}$

For a stationary field second-order coherence involves $g^{(2)}(\tau) = 1$

* For a fluctuating classical field, we may introduce a probability distribution $P(\mathcal{E})$ describing the probability of the field of having a given amplitude $\mathcal{E}$ (in general, if we have more than one mode we will have a probability $P(\mathcal{E}_k)$ of a given amplitude $\mathcal{E}_k$ for the mode k):

$$E^{+}(\mathcal{E},t) = -i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} \mathcal{E}\, e^{-i\omega t}$$

· The 2nd order correlation function may be written as (for a single mode)

$$G^{(2)}(\tau) = \int P(\mathcal{E})\, E^-(\mathcal{E},t)\, E^-(\mathcal{E},t+\tau)\, E^+(\mathcal{E},t+\tau)\, E^+(\mathcal{E},t)\, d^2\mathcal{E}$$

Then

$$g^{(2)}(0) = \frac{\int P(\mathcal{E})\, E^-(\mathcal{E},0)\, E^-(\mathcal{E},0)\, E^+(\mathcal{E},0)\, E^+(\mathcal{E},0)\, d^2\mathcal{E}}{\left|\int P(\mathcal{E})\, E^-(\mathcal{E},0)\, E^+(\mathcal{E},0)\, d^2\mathcal{E}\right|^2}$$

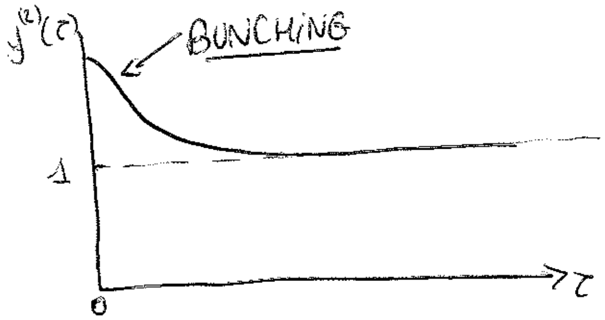
$$= \frac{\int P(\mathcal{E})\, [\,|\mathcal{E}|^4\,]\, d^2\mathcal{E}}{\left|\int P(\mathcal{E})\, |\mathcal{E}|^2 d^2\mathcal{E}\right|^2} = \frac{\int P(\mathcal{E})\, |\mathcal{E}|^4\, d^2\mathcal{E}}{\langle|\mathcal{E}|^2\rangle^2}$$

$$= 1 + \frac{1}{\langle|\mathcal{E}|^2\rangle^2} \int P(\mathcal{E}) \left[|\mathcal{E}|^4 - \langle|\mathcal{E}|^2\rangle^2\right]$$

$$= 1 + \frac{1}{\langle|\mathcal{E}|^2\rangle^2} \int P(\mathcal{E}) \left[|\mathcal{E}|^2 - \langle|\mathcal{E}|^2\rangle\right]^2$$

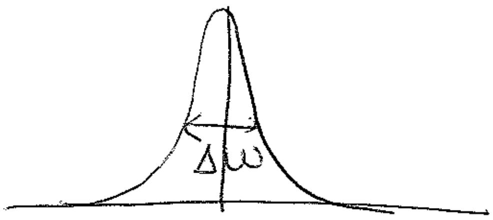
Classically $P(\mathcal{E}) \geq 0$, and hence $g^{(2)}(0) \geq 1$

· This was actually what the original Hanbury-Brown and Twiss experiment found. Although the light (coming from a star) was first-order coherent, it was not second order coherent because what they found was something like this



· If a field fluctuates there is a high probability that the photon which triggers the counter occurs during a high intensity fluctuation, and hence there's a high probability that a photon will be detected arbitrarily soon (and therefore $g^{(2)}(0) > g^{(2)}(\tau \neq 0)$) This effect is known as photon bunching

* Note that for sufficiently long times $g^{(2)}(\tau) \longrightarrow 1$.
This is because the fields are not purely monochromatic, i.e. they don't have exactly just one frequency ω but a given spectrum



* Without entering into too many details, for a Gaussian spectrum (and assuming that the field amplitude has Gaussian fluctuations with zero mean) one has something like

$$g^{(2)}(\tau) = 1 + e^{-(\tau/\tau_c)^2} \qquad \text{where } \tau_c \sim 1/\Delta\omega$$

This τ_c is the correlation time. If $\tau \gg \tau_c$ then

$g_2(\tau) \longrightarrow 1$, as we said before.

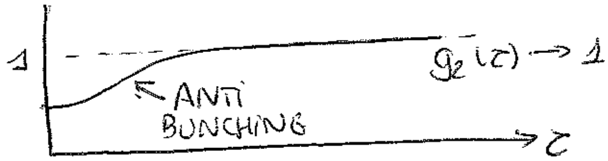
* In other words, for $\tau \gg \tau_c$ the fields become uncorrelated, and one can factorize the correlation, recovering $g_2(\tau) = 1$.

* Since $g_2(0) > 1$, then one has a bunching effect.

<u>Note</u>: since $g_2(\tau)$ is sensitive to the spectral width, one can make coincidence experiments of the Hanbury-Brown and Twiss one to determine the linewidth with very large precision.

· Please note that the previous discussion doesn't rely on any quantization of the EM field, and can be deduced from a purely classical analysis of fluctuating EM fields.

· We will see in a moment that for quantum field it's indeed possible to have $g^{(2)}(0) < 1$ (photon <u>antibunching</u>), which as we just saw it is impossible for classical fields.



* <u>QUANTUM MECHANICAL FIELDS</u>

* Let's have a look now to $g^{(2)}$ for some quantum mechanical fields. We will limit ourselves to the single-mode case.

Remember that $\hat{\mathcal{E}}^{(-)}(t) \propto \hat{a}^{\dagger}$

$\hat{\mathcal{E}}^{(+)}(t) \propto \hat{a}$

Hence:
$$\boxed{g^{(2)}(0) = \frac{\langle \hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\rangle}{\langle \hat{a}^{\dagger}\hat{a}\rangle^{2}} = 1 + \frac{V(n) - \bar{n}}{\bar{n}^{2}}}$$

where $\bar{n} = \langle \hat{a}^{\dagger}\hat{a}\rangle$

$V(n) = \langle (\hat{a}^{\dagger}\hat{a})^{2}\rangle - \langle \hat{a}^{\dagger}\hat{a}\rangle^{2}$

* For a coherent state $|\alpha\rangle$ $\langle \hat{a}^{\dagger}\hat{a}\rangle = \langle \alpha|\hat{a}^{\dagger}\hat{a}|\alpha\rangle = |\alpha|^{2} = \bar{n}$

and $V(n) = \bar{n}$ (Poissonian distribution).

Hence $\boxed{g^{(2)}(0) = 1}$

Not surprisingly a coherent state has 2nd order coherence (actually a coherent state has n^{th} order coherence).

• Let's see now what happens with a Fock state $|n\rangle$

$$\left.\begin{array}{l}\bar{n} = n \\ V(n) = 0\end{array}\right\} \text{ Then } g^{(2)}(0) = 1 - \frac{1}{n}$$

(of course here we need $n > 2$ in order to have some possibility for a coincidence measurement)

Then, $g^{(2)}(0) < 1$, contrary to the classical case.

Since, as commented before $g^{(2)}(\tau) \longrightarrow 1$ for $\tau \gg \tau_c$, then this means that if $g^{(2)}(0) < 1$ the field exhibits antibunching on some time scale.

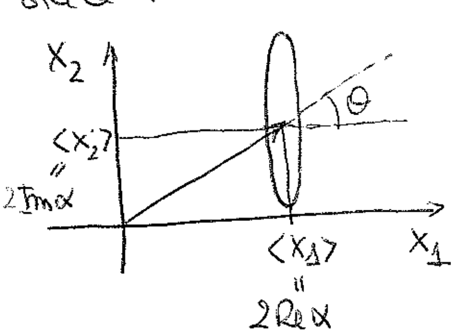
* Note that a value $g^{(2)}(0) < 1$ cannot be predicted classically, since classically $g^{(2)}(0) \geq 1$ always.

(Note: Note that ~~to~~ get $g^{(2)}(0) < 1$ one would require negative probability. This is exactly what we will introduce later in these lectures, when we comment about quasiprobability).

---

* Let's consider now a squeezed state $|\alpha, r\rangle$ (i.e. $\varepsilon = r > 0$)

Since $r > 0$, then the squeezed axis is X_1, whereas X_2 is streched.



Then $2\alpha = \langle X_1 \rangle + i \langle X_2 \rangle$

$$\alpha = |\alpha| e^{i\theta} \longrightarrow \theta = \arctan \frac{\langle X_2 \rangle}{\langle X_1 \rangle}$$

• Remember that for a squeezed state

$$V(n) = \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 = |\alpha\, \mathrm{ch}\, r - \alpha^* \mathrm{sh}\, r|^2 + 2\, \mathrm{sh}^2 r\, \mathrm{ch}^2 r$$

$$= (\mathrm{ch}\, 2r - \mathrm{sh}\, 2r \cos 2\theta)|\alpha|^2 + 2\, \mathrm{sh}^2 r\, \mathrm{ch}^2 r$$

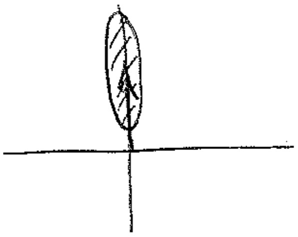
$$\bar{n} = \langle \hat{a}^\dagger \hat{a} \rangle = |\alpha|^2 + \mathrm{sh}^2 r$$

· Then:

$$\frac{V(n)-\bar{n}}{\bar{n}^2} = \frac{|\alpha|^2\left[\operatorname{ch}2r - \operatorname{sh}2r\cos 2\theta - 1\right] + \operatorname{sh}^2 r\,\operatorname{ch}2r}{\left(|\alpha|^2 + \operatorname{sh}^2 r\right)^2}$$

* If $\theta = \pi/2$ $\longrightarrow$ squeezing is out of phase with the complex amplitude

$$V(n) = |\alpha|^2 e^{2r} + 2\operatorname{sh}^2 r\,\operatorname{ch}^2 r$$

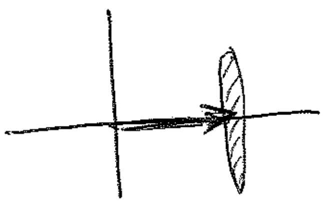


This state has increased amplitude fluctuations (remember that for a coherent state $V(n) = |\alpha|^2$)

This state has superpoissonian statistics $V(n) > \bar{n}$, and hence $g_2(0) > 1$ $\longrightarrow$ This state shows bunching.

* If $\theta = 0$ $\longrightarrow$ squeezing in phase with the complex amplitude

Then $V(n) = \underbrace{|\alpha|^2 e^{-2r}}_{\text{reduction of the number fluctuations in the original Poissonian distribution}} + \underbrace{2\sinh^2 r\,\cosh^2 r}_{\text{due to the fluctuations of the additional photons in the squeezed vacuum}}$



when $|\alpha|^2 \gg 2\operatorname{sh}^2 r\,\operatorname{ch}^2 r$ $\longrightarrow$ $V(n) < \bar{n}$ $\longrightarrow$ subpoissonian statistics.

Hence $g_2(0) < 1$, and we expect antibunching

* For a squeezed vacuum $|\alpha| = 0$, then $V(n) = 2\operatorname{sh}^2 r\,\operatorname{ch}^2 r$, and $\bar{n} = \operatorname{sh}^2 r$, hence $V(n) = 2\operatorname{ch}^2 r\,\bar{n} = (1 + \operatorname{ch}2r)\,\bar{n}$

Hence $V(n) > \bar{n}$ $\longrightarrow$ superpoissonian statistics