• REPRESENTATIONS OF THE ELECTROMAGNETIC FIELD

* A full description of the EM field demands a quantum statistical treatment, which is provided once we know the density operator $\hat{\rho}$ (also called density matrix, remember that in statistical mechanics one employs the canonical density operator $\hat{\rho} = \frac{e^{-\beta\hat{H}}}{Z}$)

NOTE: Remember that $\langle\hat{O}\rangle = \mathrm{Tr}\{\hat{\rho}\,\hat{O}\}$)

• There are different ways of representing the density operator $\hat{\rho}$ of the electromagnetic field. In this block of lectures we will analyze different possibilities, introducing rather important concepts as P-representation and Wigner-function, which will be very useful in future analyses later on.

• EXPANSION IN NUMBER STATES

• The Fock states form, as we already know, a complete set, and hence we can express a general expansion of $\hat{\rho}$:

$$\hat{\rho} = \sum_{n,m} C_{nm} \, |n\rangle\langle m|$$

The expansion in Fock states is particularly useful when only the photon number distribution is of interest. Then we can just consider the reduced expansion:

$$\hat{\rho} = \sum_{n} P_n \, |n\rangle\langle n|$$

where $P_n \equiv$ probability to have n photons in the mode.

Let's consider now a so-called <u>thermal (or chaotic) state</u>. These states describe very well the black body radiation, and hence one can describe well thermal light, like e.g. the light coming from a bulb.

* Remember the canonical density matrix $\hat{\rho} = \frac{e^{-\beta\hat{H}}}{Z}$

Now $\hat{H} = \hbar\omega\hat{a}^{+}\hat{a} = \hbar\omega\hat{n}$

and $Z = \mathrm{Tr}\{e^{-\beta\hat{H}}\} = \sum_n e^{-\beta\hbar\omega n} = (1-e^{-\beta\hbar\omega})^{-1} \equiv$ Canonical Partition function

arithmetic series

Then $\hat{\rho} = (1-e^{-\beta\hbar\omega})\, e^{-\beta\hbar\omega\hat{n}}$

$$= \sum_n (1-e^{-\beta\hbar\omega}) e^{-\beta\hbar\omega n} |n\rangle\langle n|$$

One can evaluate

arithmetic-geometric series

$$\bar{n} = \langle\hat{n}\rangle = \mathrm{Tr}\{\hat{\rho}\hat{n}\} = (1-e^{-\beta\hbar\omega}) \sum_n n e^{-\beta\hbar\omega n} \;\underline{\underline{\;=\;}}$$

$$= (1-e^{-\beta\hbar\omega}) \frac{e^{-\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2} = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} \longrightarrow e^{-\beta\hbar\omega} = \frac{\bar{n}}{1+\bar{n}}$$

Hence $\hat{\rho} = \sum_n \frac{1}{1+\bar{n}} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|$

Then, for a chaotic state $P_n = \frac{1}{(1+\bar{n})} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n$

(Note: one can easily calculate that $\langle n^2\rangle = 2\bar{n}^2 + \bar{n}$, hence $(\Delta n)^2 = \langle n^2\rangle - \langle n\rangle^2 = \bar{n}^2 + \bar{n}$. For large $\bar{n} \rightarrow \Delta n \sim \bar{n}$
The variance is as big as the mean, i.e. the fluctuations are really strong. This is why these states deserve the name chaotic!)

• P - REPRESENTATION

• As we know, the coherent states form an overcomplete set of states (remember that $\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = 1$). They may be then used as a basis set despite of the fact that (as you should remember) they are non-orthogonal. We will see now that a representation in the basis of coherent states can be very helpful in many interesting situations.

* Let's introduce the following diagonal representation of $\hat{\rho}$ in terms of coherent states:

$$\hat{\rho} = \int P(\alpha)\, |\alpha\rangle\langle\alpha|\, d^2\alpha$$

The function $\hat{P}$ is the so-called P-representation.

* Now, one can imagine $P(\alpha)$ as a sort of probability distribution for the values of α. However:

(i) The coherent states aren't orthogonal, and hence $P(\alpha)$ isn't a genuine probability distribution.
However, remember that $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$. Then if $P(\alpha)$ varies sufficiently slowly there's an approximative sense in which $P(\alpha)$ may be interpreted as a probability.

(ii) However, there are some quantum EM states for which $P(\alpha) < 0$ or it's very singular. In this case the classical "probability" picture cannot be recovered any more!

• Let's see some examples

$\ast$ Coherent state $\longrightarrow$ $\rho = |\alpha_0\rangle\langle\alpha_0| \longrightarrow P(\alpha) = \underbrace{\delta^{(2)}}_{\text{2D Dirac-Delta}}(\alpha - \alpha_0)$

* let's have a look to a <u>chaotic state</u>

For a chaotic state $P(\alpha) = \frac{1}{\pi\bar{n}} e^{-|\alpha|^2/\bar{n}}$

let's see that we recover the form of P_n we found before:

$$P_n = \langle n|\hat{\rho}|n\rangle = \int P(\alpha)\, |\langle n|\alpha\rangle|^2 d^2\alpha = \frac{1}{\pi\bar{n}}\int e^{-|\alpha|^2/\bar{n}} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} d^2\alpha$$

$$\overset{|\alpha|=r}{=} \frac{1}{\bar{n}\, n!}\int_0^\infty 2r\,dr\; e^{-r^2\left(\frac{1}{\bar{n}}+1\right)} r^{2n} \overset{r^2=y}{=}$$

$$= \frac{1}{\bar{n}\, n!}\int_0^\infty dy\; e^{-\left(\frac{1}{\bar{n}}+1\right)y} y^n \overset{\left(\frac{1+\bar{n}}{\bar{n}}\right)y = x}{=} \frac{1}{\bar{n}\left(\frac{1+\bar{n}}{\bar{n}}\right)^{n+1} n!} \underbrace{\int_0^\infty dx\, e^{-x} x^n}_{\Gamma(n+1)=n!}$$

$$= \frac{1}{(1+\bar{n})}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n$$ as we knew already.

• The P-representation is particularly useful for evaluating <u>normally-ordered products of operators</u>. For example

$$\langle \hat{a}^{+n} \hat{a}^m\rangle = \int P(\alpha)(\alpha^*)^n \alpha^m\, d^2\alpha$$

i.e. we have reduced the quantum mechanical expectation values to a sort of classical averaging (but now, once more, $P(\alpha)$ isn't a classical probability distribution). In this classical average we just have to substitute $\hat{a}^+ \longrightarrow \alpha^*$, and $\hat{a} \longrightarrow \alpha$.

• let's see the 2nd order correlation function. Remember that

$$g^{(2)}(0) = 1 + \frac{V(n)-\bar{n}}{\bar{n}^2} = 1 + \frac{[\langle(\hat{a}^+\hat{a})^2\rangle - \langle\hat{a}^+\hat{a}\rangle^2] - \langle\hat{a}^+\hat{a}\rangle}{\langle\hat{a}^+\hat{a}\rangle^2}$$

$$= 1 + \frac{[\langle(\hat{a}^+)^2(\hat{a})^2\rangle + \cancel{\langle\hat{a}^+\hat{a}\rangle} - \langle\hat{a}^+\hat{a}\rangle^2] - \cancel{\langle\hat{a}^+\hat{a}\rangle}}{\langle\hat{a}^+\hat{a}\rangle^2}$$

Now all averages are over normally ordered products of operators, and we can easily apply the P-representation.

• Hence:

$$g^{(2)}(0) = 1 + \frac{\int P(\alpha)\left[|\alpha|^4 - |\alpha|^2\right] d^2\alpha}{\left(\int P(\alpha)|\alpha|^2 d^2\alpha\right)^2} =$$

$$= 1 + \frac{\int P(\alpha)\left[|\alpha|^2 - \langle|\alpha|^2\rangle\right]^2 d^2\alpha}{\left(\int P(\alpha)|\alpha|^2 d^2\alpha\right)^2}$$

This looks formally identical as the result we got for classical fields. Remember that in the discussion about $g^{(2)}$ in classical fields we got to the conclusion that $g^{(2)}(0) > 1$ (and hence bunching) because the probability wasn't negative. Now we cannot say that, because $P(\alpha)$ may be negative, allowing $g^{(2)}(0) < 1$, and hence photon antibunching.

* Something similar occurs with squeezing.

Remember that $\hat{X}_1 = \hat{a}^+ + \hat{a}$

Hence $\langle \hat{X}_1^2 \rangle = \langle \hat{a}^{+2} + \hat{a}^2 + 2\hat{a}^+\hat{a} + 1 \rangle = 1 + \langle (\alpha^*)^2 + \alpha^2 + 2|\alpha|^2 \rangle$

$$= 1 + \langle (\alpha^* + \alpha)^2 \rangle$$

$$\Delta X_1^2 = \langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle = 1 + \langle (\alpha^* + \alpha)^2 \rangle - \langle \alpha^* + \alpha \rangle^2$$

$$= 1 + \int P(\alpha)\left[(\alpha^* + \alpha) - \langle \alpha^* + \alpha \rangle\right]^2 d^2\alpha$$

Hence the condition for squeezing $\Delta X_1 < 1$ requires a negative $P(\alpha)$, which, as commented before is classically impossible. Thus, squeezing and antibunching are purely quantum phenomena.

. Quantized field for which $P(\alpha) > 0$ do not exhibit quantum properties as antibunching or squeezing. Such fields can

be simulated by a classical description which treats the complex field amplitude $\mathcal{E}$ as a random variable with distribution $P(\mathcal{E})$. This is the case of coherent and chaotic fields.
On the contrary quantum fields exhibiting antibunching or squeezing cannot be simulated in classical terms.

- In order to compute the $P(\alpha)$ distribution, it is often convenient to introduce the characteristic function

$$\chi(\eta) = \mathrm{Tr}\{\hat{\rho}\, e^{\eta \hat{a}^{+} - \eta^{*}\hat{a}}\} \longrightarrow \text{symmetrically-ordered characteristic function}$$

and in particular the normally ordered version of it:

$$\chi_N(\eta) = \mathrm{Tr}\{\hat{\rho}\, e^{\eta \hat{a}^{+}} e^{-\eta^{*}\hat{a}}\} \longrightarrow \text{normally-ordered characteristic function}$$

From the Baker-Hausdorff formula, it's easy to see that $\chi_N(\eta) = e^{|\eta|^2/2}\chi(\eta)$

* In the P-representation:

$$\chi_N(\eta) = \int d^2\alpha\, P(\alpha)\, e^{\eta\alpha^{*}} e^{-\eta^{*}\alpha} \underset{\substack{\alpha=\alpha_r+i\alpha_i \\ \eta=\eta_r+i\eta_i}}{=}$$

$$= \int d\alpha_r\, d\alpha_i\, P(\alpha_r,\alpha_i)\, e^{i2\eta_i\alpha_r}\, e^{-i2\eta_r\alpha_i}$$

Then $\chi_N(\eta)$ is actually a 2-dimensional Fourier transform

Hence $P(\alpha)$ is the inverse Fourier transform:

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\eta\, e^{\alpha\eta^{*} - \alpha^{*}\eta}\, \chi_N(\eta)$$

- Note: Clearly: $\langle(\hat{a}^{+})^{n}(\hat{a})^{m}\rangle = \mathrm{Tr}\{\hat{\rho}\,\hat{a}^{+n}\hat{a}^{m}\} = \left[\left(\frac{\partial}{\partial\eta}\right)^{n}\left(-\frac{\partial}{\partial\eta^{*}}\right)^{m}\mathrm{Tr}\{\hat{\rho}\, e^{\eta\hat{a}^{+}} e^{-\eta^{*}\hat{a}}\}\right]_{\eta\to 0}$

$$= \left[\left(\frac{\partial}{\partial\eta}\right)^{n}\left(\frac{-\partial}{\partial\eta^{*}}\right)^{m}\chi_N(\eta)\right]_{\eta\to 0} = \left[\int d^2\alpha\, P(\alpha)\,[(\alpha^{*})^{n}(\alpha)^{m}]\, e^{\eta\alpha^{*}} e^{-\eta^{*}\alpha}\right]_{\eta\to 0}$$

$$= \int d^2\alpha\, P(\alpha)(\alpha^{*})^{n}(\alpha)^{m}$$ as we saw before.

• WIGNER REPRESENTATIONS

* By using the symmetrically-ordered characteristic function, we can define another type of representation, which turns out to be very useful, namely the Wigner representation:

$$W(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\chi(\eta)\, d^2\eta \qquad \left(\begin{array}{c}\text{Fourier-Transform of}\\ \chi(\eta)\end{array}\right)$$

- Note: $\chi(\eta) = \int d^2\alpha\, W(\alpha)\, e^{\eta\alpha^* - \eta^*\alpha}$

* We can easily obtain the relation between $W(\alpha)$ and $P(\alpha)$:

$$W(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\chi(\eta)\, d^2\eta = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*} e^{-\frac{1}{2}|\eta|^2}\chi_N(\eta)\, d^2\eta$$

$$= \frac{1}{\pi^2}\int d^2\eta\, e^{-|\eta|^2/2}\, \underbrace{\mathrm{Tr}\{\hat\rho\, e^{\eta(\hat a^\dagger - \alpha^*)} e^{-\eta^*(\hat a - \alpha)}\}}_{\int d^2\beta\, P(\beta)\, e^{\eta(\beta^* - \alpha^*)} e^{-\eta^*(\beta - \alpha)}}$$

$$= \frac{1}{\pi^2}\int d^2\eta\, d^2\beta\, P(\beta)\, \exp\left[\eta(\beta^* - \alpha^*) - \eta^*(\beta - \alpha) - \frac{1}{2}|\eta|^2\right]$$

Note: let $\lambda = \beta - \alpha \rightarrow \int d^2\eta\, e^{\lambda^*\eta} e^{-\lambda\eta^*} e^{-|\eta|^2/2} =$

$$= \int d\eta_r \int d\eta_i\, e^{(\lambda^* - \lambda)\eta_r} e^{i(\lambda^* + \lambda)\eta_i} e^{-\frac{1}{2}\eta_r^2} e^{-\frac{1}{2}\eta_i^2}$$

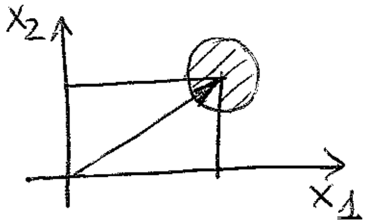
$$= \underbrace{\int_{-\infty}^{\infty} d\eta_r\, e^{(\lambda^* - \lambda)\eta_r} e^{-\eta_r^2/2}}_{\underbrace{\int_{-\infty}^{\infty} d\eta_r\, e^{-\frac{1}{2}(\eta_r - (\lambda^* - \lambda))^2}}_{\sqrt{2\pi}} e^{-\frac{1}{2}(\lambda^* - \lambda)^2}} \; \underbrace{\int_{-\infty}^{\infty} d\eta_i\, e^{i(\lambda^* + \lambda)\eta_i} e^{-\eta_i^2/2}}_{\sqrt{2\pi}\, e^{-(\lambda^* + \lambda)^2/2}}$$

$$= 2\pi\, e^{-2|\lambda|^2}$$

Then $\boxed{W(\alpha) = \frac{2}{\pi}\int d^2\beta\, P(\beta)\, e^{-2|\beta - \alpha|^2}}$

· Thus, actually, the Wigner-function is a Gaussian convolution of the P-function.

* The Wigner-function is very useful. For example, you remember that we represented the coherent state as a circle in the quadrature plane



The error area (the circle in this case) may be derived rigurously as contours of the Wigner function. We will see this immediately in an example.

· let's consider first a coherent state $|\alpha_0\rangle \rightarrow P(\alpha) = \delta^{(2)}(\alpha-\alpha_0)$

Then $W(\alpha) = \frac{2}{\pi} e^{-2|\alpha_0-\alpha|^2}$

let $|\alpha_0\rangle = |\frac{1}{2}(\bar{X}_1 + i\bar{X}_2)\rangle$; let $\alpha = \frac{1}{2}(X_1' + iX_2')$

Then $W(X_1', X_2') = \frac{2}{\pi} e^{-\frac{1}{2}[(X_1-\bar{X}_1)^2 + (X_2-\bar{X}_2)^2]} = \frac{2}{\pi} e^{-\frac{1}{2}(X_1'^2 + X_2'^2)}$

If we now consider the points at which W decays to $1/\sqrt{e}$ of it's maximal value, then this define a contour

$X_1'^2 + X_2'^2 = 1 \longrightarrow$ i.e. a circle of radius 1 centered at $(\bar{X}_1, \bar{X}_2) \Longrightarrow$ the shadowed error region in the scheme above.

* let see now what happens with a squeezed state $|\alpha, \varepsilon\rangle$

Let $\varepsilon = r$ (i.e. we consider ε as real, $\phi = 0$).

Now $\chi(\eta) = \mathrm{Tr}\{\rho\, e^{\eta\hat{a}^\dagger - \eta^*\hat{a}}\} = \langle\alpha,\varepsilon| e^{\eta\hat{a}^\dagger - \eta^*\hat{a}} |\alpha,\varepsilon\rangle$

Remember that $|\alpha,\varepsilon\rangle = \hat{D}(\alpha)\hat{S}(\varepsilon)|0\rangle$

* Using the Baker-Hausdorff formula, and the properties of the displacement operator $\hat{D}(\alpha)$ and the squeezing operator $\hat{S}(\varepsilon)$ one can obtain (exercise) that the symmetrically ordered characteristic function is:

$$\chi(\eta) = e^{\eta\alpha^* - \eta^*\alpha} \, e^{-|\eta|^2 \frac{\mathrm{ch}\,2r}{2}} \, e^{-\frac{1}{4}\mathrm{sh}\,2r\,(\eta^2+\eta^{*2})}$$

and from here one obtains (exercise) that

remember that $d^2\eta = d\eta_r \, d\eta_I$

$\eta = \eta_r + i\eta_I$

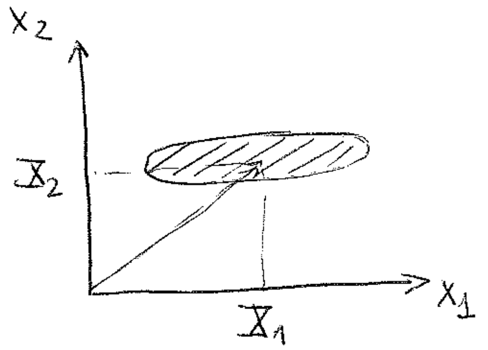
$$W(\beta) = \int \frac{d^2\eta}{\pi^2} \, e^{\eta^*\beta - \eta\beta^*} \chi(\eta) \Rightarrow$$

$$\Rightarrow W(\beta) = \frac{2}{\pi} \exp\left\{-\frac{1}{2}\left[(2\,\mathrm{Im}(\beta-\alpha))^2 e^{-2r} + (2\,\mathrm{Re}(\beta-\alpha))^2 e^{2r}\right]\right\}$$

Remember that
$$\left.\begin{aligned}\beta &= \tfrac{1}{2}(x_1 + i x_2) \\ \alpha &= \tfrac{1}{2}(X_1 + i X_2)\end{aligned}\right\} \quad \begin{aligned} 2\,\mathrm{Im}(\beta-\alpha) &= x_2' = x_2 - X_2 \\ 2\,\mathrm{Re}(\beta-\alpha) &= x_1' = x_1 - X_1\end{aligned}$$

Hence:
$$W(x_1', x_2') = \frac{2}{\pi} e^{-\frac{x_1'^2}{2e^{2r}}} \, e^{-\frac{x_2'^2}{2e^{-2r}}} = \frac{2}{\pi} e^{-\frac{1}{2}\left(\frac{x_1'^2}{e^{2r}} + \frac{x_2'^2}{e^{-2r}}\right)}$$

The contour of the Wigner function at which W decay a factor $1/\sqrt{e}$ is then given by the ellipse $\boxed{\frac{x_1'^2}{e^{2r}} + \frac{x_2'^2}{e^{-2r}} = 1}$



Therefore, we recover the same sort of picture we got previously as sketch for the squeezed state.

(Note: you can see (exercise) that a similar calculation but with χ_N should give you $P(\beta)$, but you will see that this calculation diverges. No sensible P-representation of a squeezed state exists!)

* Note:

The Wigner representation is particularly useful to evaluate correlations between quadratures, e.g.

$$\langle \hat{X}_2^n \hat{X}_1^m \rangle$$

- let's see that:

$$\chi(\eta) = \mathrm{Tr}\{\rho\, e^{\eta \hat{a}^\dagger - \eta^* \hat{a}}\} = \mathrm{Tr}\{\rho\, e^{\eta_r \hat{X}_2 + i\eta_i \hat{X}_1}\}$$

Then:

$$\langle \hat{X}_2^n \hat{X}_1^m \rangle = \left[\left(\frac{\partial}{\partial \eta_r}\right)^n \left(\frac{-i\partial}{\partial \eta_i}\right)^m \chi(\eta_r, \eta_i) \right]_{\eta \to 0}$$

Since $\chi(\eta_r, \eta_i) = \int d^2\alpha\, W(\alpha)\, e^{\eta_r(\alpha^* - \alpha)} e^{i\eta_i(\alpha^* + \alpha)}$

Then

$$\langle \hat{X}_2^n \hat{X}_1^m \rangle = \int d^2\alpha\, W(\alpha)\, (\alpha^* - \alpha)^n (\alpha^* + \alpha)^m$$

* So in order to evaluate correlations of the form $\langle \hat{X}_2^n \hat{X}_1^m \rangle$ we just have to substitute $X_2 \to \alpha^* - \alpha$

$X_1 \to \alpha^* + \alpha$

and the integrate using the Wigner function.

- Once more we have substituted operators by c-numbers which is clearly the most practical application of the quasiprobability distributions.

- Wigner Function

- The Wigner-function is rather important in quantum mechanics. In the literature you can find it in a ~~diff~~ different way, which is actually completely equivalent to the way we have ~~[illegible]~~ seen here.

- We have defined

$$W(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\chi(\eta)\, d^2\eta$$

with $\chi(\eta) = \mathrm{Tr}\{\hat{\rho}\, e^{\eta\hat{a}^+ - \eta^*\hat{a}}\}$

Remember that $\hat{a}$ and $\hat{a}^+$ can be associated to an harmonic oscillator. Taking $\omega = 1$, we can then define the position operator $\hat{x} = \frac{\sqrt{\hbar}}{\sqrt{2}}(\hat{a} + \hat{a}^+)$

and the momentum operator

$$\hat{p} = \frac{\sqrt{\hbar}\, i}{\sqrt{2}}(\hat{a} - \hat{a}^+)$$

Then $\chi(\eta) = \mathrm{Tr}\{\hat{\rho}\, e^{\eta_r(\hat{a}^+ - \hat{a}) + i\eta_i(\hat{a}^+ + \hat{a})}\} = \mathrm{Tr}\{\hat{\rho}\, e^{i\eta_r\sqrt{\frac{2}{\hbar}}\hat{p} + i\eta_i\sqrt{\frac{2}{\hbar}}\hat{x}}\}$

On the other hand

$\alpha - \alpha^* = -i\sqrt{2}\,p$

$\alpha + \alpha^* = \sqrt{2}\,x$

$$e^{\eta^*\alpha - \eta\alpha^*} = e^{\eta_r(\alpha - \alpha^*) - i\eta_i(\alpha + \alpha^*)} = e^{-i\eta_r p\sqrt{\frac{2}{\hbar}} - i\eta_i x\sqrt{\frac{2}{\hbar}}}$$

Then $W(x,p) = \frac{1}{\pi^2}\int d\eta_r\, d\eta_i\ \mathrm{Tr}\{\hat{\rho}\, e^{i\eta_r\sqrt{\frac{2}{\hbar}}(\hat{p} - p) + i\eta_i\sqrt{\frac{2}{\hbar}}(\hat{x} - x)}\}$

*

- Let $\mathrm{Tr}\,\hat{O} = \int dx' \langle x'|\hat{O}|x'\rangle$

- We employ now the Baker-Hausdorff formula

$$e^{i\eta_r\sqrt{2}(\hat{p}-p)+i\eta_i\sqrt{2}(\hat{x}-x)} = e^{i\eta_r\sqrt{\frac{2}{\hbar}}(\hat{p}-p)}\, e^{i\eta_i\sqrt{\frac{2}{\hbar}}(\hat{x}-x)}\, e^{-i\eta_r\eta_i}$$

Then

$$W(x,p) = \frac{1}{\pi^2}\int d\eta_r \int d\eta_i\, e^{-i\eta_r\eta_i} \int dx' \langle x'|\hat{\rho}\, e^{i\eta_r\sqrt{\frac{2}{\hbar}}(\hat{p}-p)}\, e^{i\eta_i\sqrt{\frac{2}{\hbar}}(x'-x)}|x'\rangle$$

$$= \frac{1}{\pi^2}\int d\eta_r \int d\eta_i\, e^{-i\eta_r\eta_i}\int dy \langle y+x|\hat{\rho}\, e^{i\eta_r\sqrt{\frac{2}{\hbar}}(\hat{p}-p)}\, e^{i\eta_i\sqrt{\frac{2}{\hbar}}y}|y+x\rangle$$

$$= \frac{1}{\pi^2}\int d\eta_r \int dy\, \underbrace{\int d\eta_i\, e^{+i\eta_i(-\eta_r+\frac{\sqrt{2}}{\sqrt{\hbar}}y)}}_{\substack{2\pi\,\delta(\sqrt{\frac{2}{\hbar}}y-\eta_r) \\ = \frac{\sqrt{\hbar}}{\sqrt{2}}2\pi\,\delta(y-\eta_r\sqrt{\frac{\hbar}{2}})}} \langle y+x|\hat{\rho}\, e^{i\eta_r\sqrt{\frac{2}{\hbar}}(\hat{p}-p)}|y+x\rangle$$

$$\xi = \eta_r\sqrt{2\hbar}$$

$$= \frac{1}{\pi^2}\int d\eta_r \frac{2\pi\sqrt{\hbar}}{\sqrt{2}} \langle x+\frac{\sqrt{\hbar}\eta_r}{\sqrt{2}}|\hat{\rho}\, e^{i\eta_r\sqrt{\frac{2}{\hbar}}(\hat{p}-p)}|x+\frac{\sqrt{\hbar}\eta_r}{\sqrt{2}}\rangle =$$

$$= \frac{1}{\pi}\int d\xi \langle x+\frac{\xi}{2}|\hat{\rho}\, e^{i\hat{p}\xi/\hbar}|x+\xi/2\rangle\, e^{-ip\xi/\hbar}$$

Remember that $e^{-i\hat{p}a}|x\rangle = |x+a\rangle$

$$= \frac{1}{\pi}\int d\xi \langle x+\xi/2|\hat{\rho}|x-\xi/2\rangle\, e^{-ip\xi/\hbar}$$

- Note: actually in the literature one uses

$$W(x,p) = \frac{1}{2\pi\hbar}\int d\xi \langle x+\xi/2|\hat{\rho}|x-\xi/2\rangle e^{-ip\xi/\hbar}$$

as the definition of the Wigner function.

This is because $dx\,dp = 2\hbar\, d\alpha_r\, d\alpha_i$

* With the Wigner function we recover in some sense an expression similar to the classical distribution function, i.e. a function of x and p (now x and p are "classical" variables, i.e. c-numbers).

* Classically the distribution function $f(x,p)$ gave the probability to find a particle with position x and momentum p.

Now $W(x,p)$ is not a probability, but as already mentioned a quasiprobability. For example whereas classically $f(x,p) \geq 0$, now $W(x,p)$ can be negative!

• Similarly as for the P-function, the Wigner function can have negative values. This is for example the case of a Fock state $|n\rangle$ for which the Wigner function acquires the form:

$$W(x_1, x_2) = \frac{2}{\pi}(-1)^n L_n[x_1^2+x_2^2]\, e^{-2(x_1^2+x_2^2)}$$

$L_n \equiv$ Laguerre Polynomial

• This Wigner function is clearly negative in this case.

$\boxed{\text{See } 65' - 65^{(IV)}}$

• THE Q-FUNCTION

* Up to now we have already seen two quasiprobability distributions, namely the P-function and the Wigner representation. Let's see now another alternative quasiprobability function which can be also useful in some situations, namely the Q-function.

$$Q(\alpha) = \frac{\langle\alpha|\rho|\alpha\rangle}{\pi} \geq 0$$

(contrary to the P- and W-functions the Q-function is always ≥ 0)

Also $Q(\alpha) \leq 1/\pi$ (this is because $Tr\,\rho = 1$)

* Similarly as for the P- and the W-functions we introduce a characteristic function (this time anti-normally ordered)

$$\chi_A(\eta) = Tr\{\hat{\rho}\, e^{-\eta^*\hat{a}}\, e^{\eta\hat{a}^+}\} \underset{\substack{\uparrow \\ \text{Cyclic property} \\ \text{of the trace}}}{=} \int \frac{d^2\alpha}{\pi} \langle\alpha|e^{\eta\hat{a}^+}\rho\, e^{-\eta^*\hat{a}}|\alpha\rangle$$

$$= \int \frac{d^2\alpha}{\pi} e^{\eta\alpha^*}\langle\alpha|\rho|\alpha\rangle e^{-\eta^*\alpha} = \int d^2\alpha\, e^{\eta\alpha^* - \eta^*\alpha}\, Q(\alpha)$$

Then the $Q(\alpha)$ function may be expressed as the Fourier transform of the antinormally ordered characteristic function:

$$Q(\alpha) = \frac{1}{\pi^2}\int e^{\alpha\eta^* - \alpha^*\eta}\, \chi_A(\eta)\, d^2\eta$$

- We can easily express the Q-function in terms of the P-function

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle = \frac{1}{\pi}\langle\alpha|\left[\int P(\beta)|\beta\rangle\langle\beta|\, d^2\beta\right]|\alpha\rangle$$

$$= \frac{1}{\pi}\int P(\beta)|\langle\alpha|\beta\rangle|^2 d^2\beta = \frac{1}{\pi}\int P(\beta)\, e^{-|\alpha-\beta|^2} d^2\beta$$

- Then, as the W-function, the Q function is a gaussian convolution of the P function (however the gaussian has $\sqrt{2}$ times the width than that for the W function)

- The Q-function is very helpful for evaluating anti-normally-ordered correlations of the form:

$$\langle \hat{a}^n \hat{a}^{+m} \rangle = \mathrm{Tr}\{\hat{\rho}\, \hat{a}^n (\hat{a}^+)^m\} = \left[\left(-\frac{\partial}{\partial\eta^*}\right)^n \left(\frac{\partial}{\partial\eta}\right)^m \mathrm{Tr}\{\hat{\rho}\, e^{-\eta^*\hat{a}} e^{\eta\hat{a}^+}\}\right]_{\eta\to 0}$$

$$= \left[\left(-\frac{\partial}{\partial\eta^*}\right)^n \left(\frac{\partial}{\partial\eta}\right)^m \chi_A(\eta)\right]_{\eta\to 0} = \left[\int d^2\alpha\, [\alpha^n(\alpha^*)^m]\, e^{\eta\alpha^*-\eta^*\alpha}\, Q(\alpha)\, d^2\alpha\right]_{\eta\to 0}$$

$$= \int d^2\alpha\, \alpha^n (\alpha^*)^m Q(\alpha)$$

Once more we have been able to convert operators into c-numbers.

* Some examples of Q-functions are:

- Number state $|n\rangle \rightarrow Q(\alpha) = \dfrac{|\langle\alpha|n\rangle|^2}{\pi} = \dfrac{|\alpha|^{2n} e^{-|\alpha|^2}}{\pi\, n!}$

- Coherent state $|\beta\rangle \rightarrow Q(\alpha) = \dfrac{|\langle\alpha|\beta\rangle|^2}{\pi} = \dfrac{e^{-|\alpha-\beta|^2}}{\pi}$

* Squeezed state $|\beta, r\rangle$

$$Q(x_1', x_2') = \frac{1}{4\pi^2 \cosh r}\, e^{-\frac{1}{2}\left[\frac{x_1'^2}{(1+e^{-2r})^2} + \frac{x_2'^2}{(1+e^{2r})^2}\right]}$$

- Thermal state with average photon number $\bar{n}$:

$$Q(\alpha) = \frac{1}{\pi(\bar{n}+1)}\, e^{-|\alpha|^2/(\bar{n}+1)}$$

* It's interesting to note that when $\bar{n} >> 1$ then for the thermal state

$$Q(\alpha) \simeq \frac{1}{\pi \bar{n}} e^{-|\alpha|^2/\bar{n}} = P(\alpha)$$

i.e. $P(\alpha)$ and $Q(\alpha)$ coincide. This is because if $\bar{n} >> 1$ then the distinction between normally-ordered and antinormally-ordered product vanish.

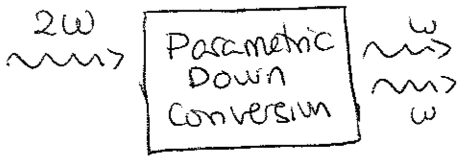
- There are other useful representations but we will not discussed them at this point.
- Summarizing, we have seen 3 types of representation
  - P-representation: $P(\alpha)$ → it simplifies the calculation of normally-ordered correlations.
  - Q-representation: $Q(\alpha)$ → it simplifies the calculation of antinormally-ordered correlations.
  - Wigner-representation: $W(\alpha)$ → it simplifies the calculation of the correlations between quadratures.
- These representations will appear rather often in future discussions.

• PARAMETRIC AMPLIFIERS

* We will now apply the formalism developed in the previous chapters to some simple processes in nonlinear optics. In particular we will see how the effects we have discussed already, as photon antibunching or squeezing may occur in nonlinear optical systems.

. Let's consider the case of a single-mode field.

. One of the simplest interactions in nonlinear optics is where a photon of frequency 2ω splits into 2 photons of frequency ω.



This process is known as parametric down conversion.
In general one may have $\omega_1 + \omega_2 = 2\omega$, i.e. 2 different modes, but we will see this case later.

. Now we are interested in the degenerate case, i.e. $2\omega \longrightarrow \omega + \omega$

. In the following we consider a simple model where the pump mode at frequency 2ω is classical and the signal mode (at frequency ω) is described by the creation/annihilation operators $\hat{a}^\dagger, \hat{a}$

The Hamiltonian describing the system is then

$$\hat{H} = \underbrace{\hbar\omega\, \hat{a}^\dagger \hat{a}}_{\omega\text{-field}} - i\hbar \frac{\chi}{2} \left(a^2 e^{2i\omega t} - \hat{a}^{\dagger 2} e^{-2i\omega t} \right)$$

we destroy 2 photons to create a 2ω one (parametric up-conversion)

we create 2 photons of freq. ω (parametric down conversion)

(Note: actually $\hat{H}$ is already in the interaction picture with respect to the classical 2ω-field) $\longrightarrow$ This is why we have $e^{\pm 2i\omega t}$ up there.

- Let's work in the interaction picture (with respect to $\hbar\omega\hat{a}^\dagger\hat{a}$)

$$\hat{H}_I = -i\hbar\frac{\chi}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})$$

(Note: the evolution operator $e^{-i\hat{H}t/\hbar} = e^{\frac{\chi t}{2}(\hat{a}^2-\hat{a}^{\dagger 2})}$
i.e. $e^{-i\hat{H}t/\hbar} = \hat{S}(\chi t) \equiv$ squeezing operator $\to$ so we expect squeezing!)

- The Heisenberg equations of motion for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ are:

$$\left.\begin{aligned}\frac{d\hat{a}}{dt} &= -\frac{i}{\hbar}[\hat{a},\hat{H}_I] = \chi\hat{a}^\dagger \\ \frac{d\hat{a}^\dagger}{dt} &= -\frac{i}{\hbar}[\hat{a}^\dagger,\hat{H}_I] = \chi a\end{aligned}\right\} \frac{d^2}{dt}\hat{a} = \chi^2\hat{a}$$

Then $\hat{a}(t) = \hat{c}_1\cosh\chi t + \hat{c}_2\sinh\chi t$

$$\left.\begin{aligned}\text{with}\quad &\hat{a}(0) = \hat{c}_1 \\ &\left(\frac{d\hat{a}}{dt}\right)_0 = \chi\hat{c}_2 = \chi\hat{a}^\dagger(0)\end{aligned}\right\}\ \boxed{\hat{a}(t) = \hat{a}(0)\,\mathrm{ch}\,\chi t + \hat{a}^\dagger(0)\,\mathrm{sh}\,\chi t}$$

* Remember that we can introduce the quadratures

$$\hat{X}_1 = \hat{a}^\dagger + \hat{a}$$
$$\hat{X}_2 = i(\hat{a}^\dagger - \hat{a})$$

Then: $\left.\begin{aligned}\hat{X}_1(t) &= e^{\chi t}\hat{X}_1(0) \\ \hat{X}_2(t) &= e^{-\chi t}\hat{X}_2(0)\end{aligned}\right\} \longrightarrow$ squeezing!

Therefore the parametric amplifier reduces the noise in one quadrature ($\hat{X}_2$) and increases it in the other ($\hat{X}_1$)

- The variances satisfy

$$V(X_1,t) = e^{2\chi t}\,V(X_1,0)$$
$$V(X_2,t) = e^{-2\chi t}\,V(X_2,0)$$

- If the state is initially a vacuum or a coherent state then $V(X_i, 0) = 1$, and hence

$$V(X_1, t) = e^{2\chi t}$$
$$V(X_2, t) = e^{-2\chi t}$$

- Note that the amount of squeezing is proportional to χt i.e. ~~it grows~~ with
  - The strength of the nonlinearity
  - The pump intensity
  - the interaction time

$\left.\begin{array}{l}\text{The strength of the nonlinearity} \\ \text{The pump intensity}\end{array}\right\}$ roughly $\chi \sim \text{strength} \times \sqrt{\text{Intensity}}$

- OK, so now we have seen that the parametric down conversion leads to squeezing. Let's have a look to another quantity we already know, namely the $g^{(2)}(0)$ correlation function.

$$g^{(2)}(0) = \frac{\langle \hat{a}^\dagger(t)\, \hat{a}^\dagger(t)\, \hat{a}(t)\, \hat{a}(t) \rangle}{\langle \hat{a}^\dagger(t)\, \hat{a}(t) \rangle^2} = 1 + \frac{\cosh 2\chi t}{\sinh^2 \chi t} \quad \left(\begin{array}{c}\text{for an initial} \\ \text{vacuum} \\ \text{state}\end{array}\right)$$

(Note that $g^{(2)}(0)$ depends on the time t, this is because the field is not stationary, obviously because we are generating photons!)

- Therefore the squeezed light generated from an initial vacuum exhibits photon bunching, since $g^{(2)}(0) > 1$

This is not so surprising because we are creating the photons in pairs!

* Let's see now what happens for ~~a~~ an initial coherent state $|\alpha\rangle$.

* One can see (exercise) that

$$g^{(2)}(0) = 1 + \frac{|\alpha|^2\left[\mathrm{ch}\,4\chi t + \cos 2\theta\,\mathrm{sh}\,4\chi t - \mathrm{ch}\,2\chi t - \cos 2\theta\,\mathrm{sh}\,2\chi t\right] + \frac{\mathrm{sh}^2 2\chi t}{2} + \mathrm{sh}^2\chi t}{\left[|\alpha|^2\left[\mathrm{ch}\,2\chi t + \mathrm{sh}\,2\chi t\cos 2\theta\right] + \mathrm{sh}^2\chi t\right]^2}$$

If $|\alpha|^2 \gg \mathrm{sh}^2\chi t,\ \mathrm{sh}\chi t\,\mathrm{ch}\chi t$

$$g^{(2)}(0) \simeq 1 + \frac{1}{|\alpha|^2}\,\frac{\left[\left[\mathrm{ch}\,4\chi t - \mathrm{ch}\,2\chi t\right] + \cos 2\theta\left[\mathrm{sh}\,4\chi t - \mathrm{sh}\,2\chi t\right]\right]}{\left[\mathrm{ch}\,2\chi t + \cos 2\theta\,\mathrm{sh}\,2\chi t\right]^2}$$

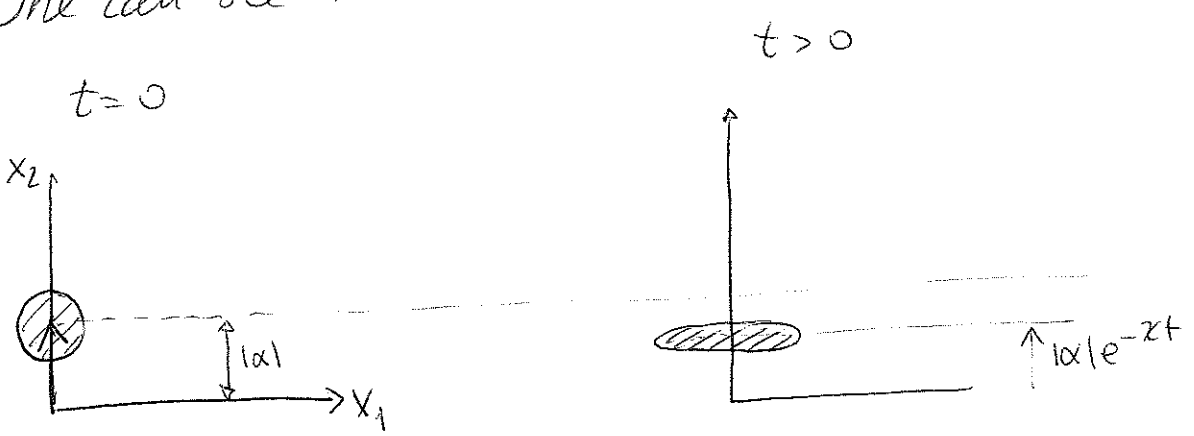
Note that for $\theta = \pi/2$

$$g^{(2)}(0) \simeq 1 + \frac{1}{|\alpha|^2}\,\frac{(\mathrm{ch}\,4\chi t - \mathrm{ch}\,2\chi t) - (\mathrm{sh}\,4\chi t - \mathrm{sh}\,2\chi t)}{\left[\mathrm{ch}\,2\chi t - \mathrm{sh}\,2\chi t\right]}$$

$$= 1 + \frac{1}{|\alpha|^2}\,\frac{e^{-4\chi t} - e^{-2\chi t}}{e^{-4\chi t}} = 1 + \frac{1}{|\alpha|^2}\left[1 - e^{2\chi t}\right] < 1$$

i.e. for some values of θ the photon statistics of the output light shows antibunching

* One can see that for $\theta = \pi/2$



One squeezes but also the center is displaced.

* Let's evaluate now the Wigner function describing the state of the parametric oscillator.

We employ the symmetrically-ordered characteristic function:

$$\chi(\eta,t) = \mathrm{Tr}\{\hat{\rho}(0)\, e^{\eta \hat{a}^\dagger(t) - \eta^* \hat{a}(t)}\}$$

* Let's consider an initial coherent state $\hat{\rho}(0) = |\alpha_0\rangle\langle\alpha_0|$

Remember that

$$\hat{a}(t) = \hat{a}(0)\,\mathrm{ch}\,\chi t + \hat{a}^\dagger(0)\,\mathrm{sh}\,\chi t$$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)\,\mathrm{ch}\,\chi t + \hat{a}(0)\,\mathrm{sh}\,\chi t$$

then:

$$\chi(\eta,t) = \langle\alpha_0| e^{\eta(\mathrm{ch}\chi t\,\hat{a}^\dagger + \mathrm{sh}\chi t\,\hat{a}) - \eta^*(\mathrm{ch}\chi t\,\hat{a} + \mathrm{sh}\chi t\,\hat{a}^\dagger)} |\alpha_0\rangle$$

$$\eta(t) = \eta\,\mathrm{ch}\chi t - \eta^*\,\mathrm{sh}\chi t$$

$$= \langle\alpha_0| e^{\eta(t)\hat{a}^\dagger - \eta^*(t)\hat{a}} |\alpha_0\rangle =$$

$$= \langle\alpha_0| e^{\eta(t)\hat{a}^\dagger} e^{-\eta^*(t)\hat{a}} e^{-\frac{1}{2}|\eta(t)|^2} |\alpha_0\rangle$$

$$= e^{-\frac{1}{2}|\eta(t)|^2} e^{\eta(t)\alpha_0^*} e^{-\eta^*\alpha_0}$$

$$= e^{-\frac{1}{2}|\eta|^2 \mathrm{ch}2\chi t}\, e^{\frac{1}{4}(\eta^2+\eta^{*2})\mathrm{sh}2\chi t}\, e^{\eta\alpha_0^*(t)}\, e^{-\eta^*\alpha_0(t)}$$

where $\alpha_0(t) = \mathrm{ch}\chi t\,\alpha_0 + \mathrm{sh}\chi t\,\alpha_0^*$

* Then:

$$\chi(\eta_r,\eta_i,t) = e^{-\frac{1}{2}(\eta_r^2+\eta_i^2)\mathrm{ch}2\chi t}\, e^{\frac{1}{2}(\eta_r^2-\eta_i^2)\mathrm{sh}2\chi t}\, e^{-i\eta_r 2\alpha_{0i}}\, e^{i\eta_i 2\alpha_{0r}}$$

$$= e^{-\frac{1}{2}\eta_r^2(\mathrm{ch}2\chi t - \mathrm{sh}2\chi t)}\, e^{-\frac{1}{2}\eta_i^2(\mathrm{ch}2\chi t + \mathrm{sh}2\chi t)}\, e^{-i\eta_r 2\alpha_{0i}}\, e^{+i\eta_i 2\alpha_{0r}}$$

Then, the Wigner function is:

$$W(\beta,t) = \int \frac{d^2\eta}{\pi^2}\; e^{\eta^*\beta - \eta\beta^*}\,\chi(\eta,t)$$

$$= \frac{1}{\pi^2}\int d\eta_r\, d\eta_i\; e^{-i\eta_i i 2\beta_{0r}}\, e^{i\eta_r 2\beta_{0i}}\,\chi(\eta,t)$$

$$= \frac{1}{\cancel{\pi}}\left[\int_{-\infty}^{\infty} d\eta_r\; e^{-\frac{1}{2}(\mathrm{ch}2\chi t - \mathrm{sh}2\chi t)\eta_r^2}\; e^{-i\eta_r 2(\alpha_{0i}-\beta_{0i})}\right]$$

$$\times\; \frac{1}{\pi}\left[\int_{-\infty}^{\infty} d\eta_i\; e^{-\frac{1}{2}(\mathrm{ch}2\chi t + \mathrm{sh}2\chi t)\eta_i^2}\, e^{+i\eta_i 2(\alpha_{0r}-\beta_{0r})}\right] =$$

$\cancel{X}_1 = \beta^* + \beta$

$X_{01}(t) = \alpha_0^*(t) + \alpha_0(t)$

$$= \frac{\pi}{\cancel{2\pi}} \exp\left\{ e^{-\frac{1}{2}\frac{(X_1 - X_{01}(t))^2}{e^{2\chi t}}} + e^{-\frac{1}{2}\left(\frac{X_2 - X_{02}(t)}{e^{-2\chi t}}\right)^2} \right\}$$

And we retrieve as expected a Gaussian whose widths are time dependent: