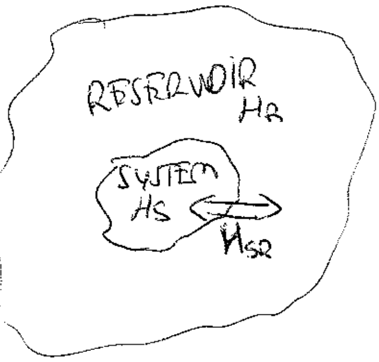
• STOCHASTIC METHODS

* In all real physical processes there's an associated loss mechanism. In this block of lectures we will see how these losses can be included in the quantum mechanical description.

· We will first derive the so-called Master equation for the density operator of the system. Then we will employ the quasiprobability distributions that we have already introduced, to transform the master equation (which is an operator equation) into a c-number equation (known as the Fokker-Plank equation).

* MASTER EQUATION

· In the following we consider a system (S) described by a Hamiltonian ($\hat{H}_S$) coupled to a reservoir (R) described by a Hamiltonian ($\hat{H}_R$). There's a weak interaction between S and R given by the Hamiltonian ($\hat{H}_{SR}$).



· We assume the reservoir as very large (it won't be affected by the system)

· The total Hamiltonian is then

$$\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_{SR}$$

Let $\hat{w}(t)$ be the total density operator of S+R in the interaction picture. The equation of motion for $\hat{w}(t)$ is then:

$$\frac{d\hat{w}(t)}{dt} = -\frac{i}{\hbar}\left[\hat{H}_{SR}(t), \hat{w}(t)\right] \qquad \left(\begin{array}{l} \hat{w}(t) = e^{i(H_S+H_R)t/\hbar}\, \hat{w}_S(t)\, e^{-i(H_S+H_R)t} \\ \hat{H}_{SR}(t) = e^{i(H_S+H_R)t/\hbar}\, \hat{H}_{SR}\, e^{-i(H_S+H_R)t} \end{array} \right)$$

· We define the reduced density operator of the system as

$$\hat{\rho}(t) = \mathrm{Tr}_R\left\{\hat{w}(t)\right\}$$

where Tr_R denotes a trace over reservoir variables
(<u>Note</u>: we will be interested in the properties of the system, so physically the trace over the reservoir variables means that we do not really care about the reservoir, but only on how it affects the system).

* Let's integrate the equation of motion:

$$\hat{W}(t) = \hat{W}(0) - \frac{i}{\hbar}\int_0^t dt' \left[\hat{H}_{SR}(t'), \hat{W}(t')\right]$$

Let's substitute this into the commutator of the equation of motion:

$$\frac{d\hat{W}}{dt}(t) = -\frac{i}{\hbar}\left[\hat{H}_{SR}(t), \hat{W}(0)\right] - \frac{1}{\hbar^2}\int_0^t dt' \left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}(t'), \hat{W}(t')\right]\right]$$

· At the moment this equation is exact. We will now introduce some useful and reasonable approximations.

We will assume that the interaction is turned on at $t=0$ and that no correlation exists between S and R at this time. Then

$$\hat{W}(0) = \hat{\rho}(0) \otimes \hat{R}_0$$

where $\hat{R}_0$ is an initial reservoir density operator.

Then, noting that $tr_R(\hat{W}(t)) = \hat{\rho}(t)$, we obtain

$$\frac{d\hat{\rho}}{dt}(t) = -\frac{i}{\hbar} tr_R\left\{\left[\hat{H}_{SR}(t), \hat{\rho}(0)\otimes R_0\right]\right\} - \frac{1}{\hbar^2}\int_0^t dt'\, tr_R\left\{\left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}(t'), \hat{W}(t')\right]\right]\right\}$$

· If the reservoir operators coupling to S have zero mean in $\hat{R}_0$
<u>note</u>: this can always be arranged by including $tr_R(\hat{H}_{SR}\hat{R}_0)$ in the system Hamiltonian)
then $Tr_R[\hat{H}_{SR}(t), \hat{R}_0] = 0$, and we arrive to the <u>master equation</u>

$$\frac{d\hat{\rho}}{dt}(t) = -\frac{1}{\hbar^2}\int_0^t dt'\, tr_R\left\{\left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}(t'), \hat{W}(t')\right]\right]\right\}$$

· We have stated that $\hat{W}(t)$ factorizes as $\hat{\rho}(0)\otimes\hat{R}_0$ at $t=0$. At later times correlations between S and R will arise due to the coupling between them. We have assumed, however, that this coupling is very weak. In addition, R is a large reservoir whose state should be basically unaffected by its couplings to S.

Hence $\hat{W}(t) \simeq \hat{\rho}(t) \otimes R_0$ (this is good up to deviations of order $\mathcal{O}(\hat{H}_{SR})$)

- Then, neglecting terms higher than second order in $\hat{H}_{SR}$ we get

$$\frac{d\hat{\rho}}{dt}(t) = -\frac{1}{\hbar^2}\int_0^t dt' \, tr_R\left\{\left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}(t'), \hat{\rho}(t') \otimes R_0\right]\right]\right\} \Leftarrow$$ This is the master Eq. in the so-called Born approximation

- This equation is however still complicated, because $\hat{\rho}(t)$ depends on its past history (i.e. on $\hat{\rho}(t')$ with $t'<t$). A second major approximation, known as Markov approximation, replaces $\hat{\rho}(t')$ by $\hat{\rho}(t)$ to obtain the master equation in the Born-Markov approximation:

$$\boxed{\frac{d\hat{\rho}}{dt} = -\frac{1}{\hbar^2}\int_0^t dt' \, tr_R\left\{\left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}(t'), \hat{\rho}(t) \otimes R_0\right]\right]\right\}}$$

- Although the Born approximation seems intuitively reasonable, the Markov approximation deserves some careful discussion. In principle S can depend on its past history because its early states become imprinted as changes in the reservoir state through the interaction $\hat{H}_{SR}$. Then earlier states are then "reflected back" on the future evolution of S as it interacts with the changed reservoir.

- However, R is a large reservoir in equilibrium, so we do not expect it to preserve the minor changes brought by the interaction with S for very long, in particular not for time scales at which S can change significantly.

Let's see this more clearly. Let's consider a general form for $\hat{H}_{SR}$

$$\hat{H}_{SR} = \hbar \sum_i \hat{s}_i \hat{\Gamma}_i \implies \text{interaction picture} \quad \hat{s}_i(t) = e^{i\hat{H}_S t/\hbar}\hat{s}_i e^{-i\hat{H}_S t/\hbar}, \quad \hat{\Gamma}_i(t) = e^{i\hat{H}_R t/\hbar}\hat{\Gamma}_i e^{-i\hat{H}_R t/\hbar}$$

$\hat{s}_i$: system operators; $\hat{\Gamma}_i$: reservoir operators

$$\Downarrow$$

$$\hat{H}_{SR}(t) = \hbar \sum_i \hat{s}_i(t)\,\hat{\Gamma}_i(t)$$

* The master equation in the Born approximation becomes:

$$\frac{d\hat{\rho}}{dt} = -\sum_{ij}\int_0^t dt'\, \mathrm{tr}_R\left\{\left[\hat{S}_i(t)\hat{\Gamma}_i(t), \left[\hat{S}_j(t')\hat{\Gamma}_j(t'), \hat{\rho}(t')\otimes\hat{R}_0\right]\right]\right\}$$

$$= -\sum_{ij}\int_0^t dt' \left[\begin{array}{l} \hat{S}_i(t)\hat{S}_j(t')\hat{\rho}(t')\,\mathrm{tr}_R\left[\hat{\Gamma}_i(t)\hat{\Gamma}_j(t')\hat{R}_0\right] \\ -\hat{S}_i(t)\hat{\rho}(t')\hat{S}_j(t')\,\mathrm{tr}_R\left[\hat{\Gamma}_i(t)\hat{R}_0\hat{\Gamma}_j(t')\right] \\ -\hat{S}_j(t')\hat{\rho}(t')\hat{S}_i(t)\,\mathrm{tr}_R\left[\hat{\Gamma}_j(t')\hat{R}_0\hat{\Gamma}_i(t)\right] \\ +\hat{\rho}(t')\hat{S}_j(t')\hat{S}_i(t)\,\mathrm{tr}_R\left[\hat{R}_0\hat{\Gamma}_j(t')\hat{\Gamma}_i(t)\right] \end{array}\right]$$

cyclic property of the Trace

$$\langle\hat{\Gamma}_i(t)\hat{\Gamma}_j(t')\rangle_R = \mathrm{Tr}_R\left[\hat{R}_0\hat{\Gamma}_i(t)\hat{\Gamma}_j(t')\right]$$

$$= -\sum_{ij}\int_0^t dt' \left[\begin{array}{l} \left[\hat{S}_i(t)\hat{S}_j(t')\hat{\rho}(t') - \hat{S}_j(t')\hat{\rho}(t')\hat{S}_i(t)\right]\langle\hat{\Gamma}_i(t)\hat{\Gamma}_j(t')\rangle_R \\ +\left[\hat{\rho}(t')\hat{S}_j(t')\hat{S}_i(t) - \hat{S}_i(t)\hat{\rho}(t')\hat{S}_j(t')\right]\langle\hat{\Gamma}_j(t')\hat{\Gamma}_i(t)\rangle_R \end{array}\right]$$

The properties of the reservoir R enter through the two correlation functions. If these correlation functions decay very rapidly in the time scale at which $\hat{\rho}(t)$ varies, then we can safely assume $\hat{\rho}(t') \simeq \hat{\rho}(t)$, i.e. the Markov approximation.

* We will look explicitly at these different time scales in the discussion of our first example of dissipative systems, namely <u>the damped harmonic oscillator</u>.

• THE DAMPED HARMONIC OSCILLATOR

* In the following we consider that our system of interest is an harmonic oscillator with frequency ω_0 and creation and annihilation operators $\hat{a}^+$ and $\hat{a}$:

$$\hat{H}_S = \hbar\omega_0 \hat{a}^+\hat{a}$$

. The reservoir R is modeled as a collection of harmonic oscillators with frequencies ω_j, and corresponding creation and annihilation operators $\hat{r}_j^+$ and $\hat{r}_j$:

$$\hat{H}_R = \sum_j \hbar\omega_j \hat{r}_j^+ \hat{r}_j$$

. The oscillator $\hat{a}$ and the j^{th} reservoir oscillator are coupled via a coupling constant k_j:

$$\hat{H}_{SR} = \sum_j \hbar\,(k_j^* \hat{a}\hat{r}_j^+ + k_j \hat{a}^+\hat{r}_j)$$

(where we implicitely assume the rotating-wave approximation)

let $\hat{\Gamma} = \sum_j k_j r_j$, then

$$\hat{H}_{SR} = \hbar\,(\hat{a}\hat{\Gamma}^+ + \hat{a}^+\hat{\Gamma})$$

* We will consider the reservoir to be in thermal equilibrium at temperature T. Hence the density operator of the reservoir is given by the canonical density operator:

$$\hat{R}_0 = \frac{e^{-\beta\hat{H}}}{Z} = \prod_j \left(1-e^{-\hbar\omega_j/k_BT}\right) e^{-\hbar\omega_j \hat{r}_j^+\hat{r}_j/k_BT}$$

. In the interaction picture

$$\hat{a}(t) = \hat{a}\,e^{-i\omega_0 t} \longrightarrow$$ This plays the role of $\hat{s}_1(t)$ in the previous discussion.

$$\hat{a}^+(t) = \hat{a}^+ e^{i\omega_0 t} \longrightarrow \hat{s}_2(t)$$

$$\hat{\Gamma}(t) = \sum_j k_j \hat{r}_j e^{-i\omega_j t} \longrightarrow \hat{\Gamma}_2(t)$$

$$\hat{\Gamma}^+(t) = \sum_j k_j^* \hat{r}_j^+ e^{i\omega_j t} \longrightarrow \hat{\Gamma}_1(t)$$

(Note that $\langle\hat{\Gamma}(t)\rangle_R = \langle\hat{\Gamma}^+(t)\rangle_R = 0$)

* Then, using our previous discussion for $\hat{H}_{SR} = \hbar \sum_i \hat{s}_i \hat{r}_i$, we obtain a master equation with 16 terms (note that we have $\sum_{i=1,2} \sum_{j=1,2}$ and four terms each):

$$\frac{d\hat{\rho}}{dt} = -\int_0^t dt' \Big\{ \left[\hat{a}\hat{a}\hat{\rho}(t') - \hat{a}\hat{\rho}(t')\hat{a}\right] e^{-i\omega_0(t+t')} \langle \hat{r}^\dagger(t)\hat{r}^\dagger(t')\rangle_R + h.c.$$
$$+ \left[\hat{a}^\dagger\hat{a}^\dagger\hat{\rho}(t') - \hat{a}^\dagger\hat{\rho}(t')\hat{a}^\dagger\right] e^{i\omega_0(t+t')} \langle \hat{r}(t)\hat{r}(t')\rangle_R + h.c.$$
$$+ \left[\hat{a}\hat{a}^\dagger\hat{\rho}(t') - \hat{a}^\dagger\hat{\rho}(t')\hat{a}\right] e^{-i\omega_0(t-t')} \langle \hat{r}^\dagger(t)\hat{r}(t')\rangle_R + h.c.$$
$$+ \left[\hat{a}^\dagger\hat{a}\hat{\rho}(t') - \hat{a}\hat{\rho}(t')\hat{a}^\dagger\right] e^{i\omega_0(t-t')} \langle \hat{r}(t)\hat{r}^\dagger(t')\rangle_R + h.c. \Big\}$$

* let's see the reservoir correlations:

$$\langle \hat{r}^\dagger(t)\hat{r}^\dagger(t')\rangle_R = \sum_{jk} \kappa_j^* \kappa_k^* e^{i\omega_j t} e^{i\omega_k t'} \mathrm{tr}_R\{\hat{R}_0 \hat{r}_j^\dagger \hat{r}_k^\dagger\} = 0$$

$$\langle \hat{r}(t)\hat{r}(t')\rangle_R = \sum_{jk} \kappa_j \kappa_k e^{-i\omega_j t} e^{-i\omega_k t'} \mathrm{tr}_R\{\hat{R}_0 \hat{r}_j \hat{r}_k\} = 0$$

$$\langle \hat{r}^\dagger(t)\hat{r}(t')\rangle_R = \sum_{jk} \kappa_j^* \kappa_k e^{i\omega_j t} e^{-i\omega_k t'} \mathrm{tr}_R\{\hat{R}_0 \hat{r}_j^\dagger \hat{r}_k\} = \quad \bar{n}(\omega_j,T) = \frac{e^{-\hbar\omega_j/k_BT}}{1-e^{-\hbar\omega_j/k_BT}}$$
$$= \sum_j |\kappa_j|^2 e^{i\omega_j(t-t')} \bar{n}(\omega_j,T)$$

$$\langle \hat{r}(t)\hat{r}^\dagger(t')\rangle_R = \sum_{jk} \kappa_j \kappa_k^* e^{-i\omega_j t} e^{i\omega_k t'} \mathrm{tr}_R\{\hat{R}_0 \hat{r}_j \hat{r}_k^\dagger\} =$$
$$= \sum_j |\kappa_j|^2 e^{-i\omega_j(t-t')} \left[\bar{n}(\omega_j,T) + 1\right]$$

- The nonvanishing reservoir correlation functions involve a summation $\sum_j$.
We change this summation into an integral

$$\sum_j \longrightarrow \int g(\omega)d\omega$$
where $g(\omega)$ = density of states (number of oscillators with frequencies between ω and $\omega+d\omega$)

then:

$$\langle \hat{r}^\dagger(t)\hat{r}(t-\tau)\rangle_R = \int_0^\infty d\omega\, e^{i\omega\tau} g(\omega) |\kappa(\omega)|^2 \bar{n}(\omega,T)$$

$$\langle \hat{r}(t)\hat{r}^\dagger(t-\tau)\rangle_R = \int_0^\infty d\omega\, e^{-i\omega\tau} g(\omega) |\kappa(\omega)|^2 \left(\bar{n}(\omega,T) + 1\right)$$

with $\bar{n}(\omega,T) = \dfrac{e^{-\hbar\omega/K_BT}}{1-e^{-\hbar\omega/K_BT}}$

Then, the master equation is of the form:

$$\frac{d\hat{\rho}}{dt} = -\int_0^t d\tau \left\{ \begin{array}{l} [\hat{a}\hat{a}^+\hat{\rho}(t-\tau) - \hat{a}^+\hat{\rho}(t-\tau)\hat{a}]e^{-i\omega_0\tau}\langle\hat{\Gamma}^+(t)\hat{\Gamma}(t-\tau)\rangle_R + h.c. \\ + [\hat{a}^+\hat{a}\,\hat{\rho}(t-\tau) - \hat{a}\,\hat{\rho}(t-\tau)\hat{a}^+]e^{i\omega_0\tau}\langle\hat{\Gamma}(t)\,\hat{\Gamma}^+(t-\tau)\rangle_R + h.c. \end{array} \right\}$$

* The correlation functions $\langle\Gamma^+(t)\,\Gamma(t-\tau)\rangle_R$ are peaked about $\tau=0$ with width $t_R = \hbar/K_BT$ (which acts as a thermal time scale for the decay of correlations). At room temperature $\hbar/K_BT \approx 10^{-13}$ s, which is typically very small in comparison with the typical time scales of variation of $\hat{\rho}$.

* As a consequence we can safely employ the Markov approximation $\hat{\rho}(t-\tau) \simeq \hat{\rho}(t)$

Then, let

$$\alpha \equiv \int_0^t d\tau \int_0^\infty d\omega\, e^{-i(\omega-\omega_0)\tau} g(\omega)\,|K(\omega)|^2$$

$$\beta \equiv \int_0^t d\tau \int_0^\infty d\omega\, e^{-i(\omega-\omega_0)\tau} g(\omega)\,|K(\omega)|^2\,\bar{n}(\omega,T)$$

Then:

$$\frac{d\hat{\rho}}{dt}(t) = \alpha\left[\hat{a}\,\hat{\rho}(t)\hat{a}^+ - \hat{a}^+\hat{a}\,\hat{\rho}(t)\right] + \beta\left[\begin{array}{l}\hat{a}\,\hat{\rho}(t)\hat{a}^+ + \hat{a}^+\hat{\rho}(t)\hat{a} - \hat{a}^+\hat{a}\hat{\rho}(t) \\ -\hat{\rho}(t)\hat{a}\hat{a}^+\end{array}\right]$$

$$+ h.c.$$

* Since, as commented above the typical time scales for $\hat{\rho}(t)$ are much larger than the correlation time in the reservoir t_R, then we can take $\int_0^t d\tau \longrightarrow \int_0^\infty d\tau$ in α and β.

* **NOTE** :

$$\int_0^t d\tau\, e^{-i(\omega-\omega_0)\tau} = \frac{\sin(\omega-\omega_0)t}{\omega-\omega_0} - i\,\frac{[1-\cos(\omega-\omega_0)t]}{\omega-\omega_0}$$

Then:

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega \int_0^t d\tau\, e^{-i(\omega-\omega_0)\tau} f(\omega) = \lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega\, f(\omega)\frac{\sin(\omega-\omega_0)t}{\omega-\omega_0} - i\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega\, f(\omega)\frac{\cos(\omega-\omega_0)t}{\omega-\omega_0}$$

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega\, f(\omega)\frac{\sin(\omega-\omega_0)t}{\omega-\omega_0} = f(\omega_0)\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega\,\frac{\sin(\omega-\omega_0)t}{\omega-\omega_0} = \pi f(\omega_0)$$

$$= \int_{-\infty}^{\infty} d\omega\, \pi\,\delta(\omega-\omega_0) f(\omega)$$

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega\, f(\omega)\frac{1-\cos(\omega-\omega_0)t}{\omega-\omega_0} = \int_{-\infty}^{\infty} d\omega\,\frac{f(\omega)}{\omega-\omega_0} - \underbrace{\lim_{t\to\infty}\int_{-\infty}^{\infty} d\omega\,\frac{f(\omega)\cos(\omega-\omega_0)t}{\omega-\omega_0}}_{\text{substracts the singularity at } \omega=\omega_0}$$

$$= P\int_{-\infty}^{\infty} d\omega\,\frac{f(\omega)}{\omega-\omega_0}$$

(remember that $P\int_{-\infty}^{\infty}\frac{f(\omega)}{\omega} = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} d\omega\frac{f(\omega)}{\omega} + \int_{+\epsilon}^{\infty} d\omega\frac{f(\omega)}{\omega}\right]$

Hence $\lim_{\tau\to\infty}\int_0^\tau d\tau\, e^{-i(\omega-\omega_0)\tau} = \pi\delta(\omega-\omega_0) + i\,\frac{P}{\omega_0-\omega}$

* We will employ the following relation:

$$\lim_{t\to\infty} \int_0^t d\tau\, e^{-i(\omega-\omega_0)\tau} = \pi\,\delta(\omega-\omega_0) + i\,\frac{P}{\omega_0-\omega}$$

see 81'

P ≡ Cauchy principal value

Hence

$$\alpha = \pi g(\omega_0)|\kappa(\omega_0)|^2 + i\Delta$$

$$\beta = \pi g(\omega_0)\,|\kappa(\omega_0)|^2\,\bar{n}(\omega_0,T) + i\Delta'$$

with $$\Delta \equiv P\int_0^\infty d\omega\,\frac{g(\omega)\,|\kappa(\omega)|^2}{\omega_0-\omega}$$

$$\Delta' \equiv P\int_0^\infty d\omega\,\frac{g(\omega)|\kappa(\omega)|^2}{\omega_0-\omega}\,\bar{n}(\omega,T)$$

let $\gamma \equiv 2\pi g(\omega_0)\,|\kappa(\omega_0)|^2$

$\bar{n} \equiv \bar{n}(\omega_0,T)$

Then $\alpha = \frac{\gamma}{2} + i\Delta$

$\beta = \frac{\gamma}{2}\bar{n} + i\Delta'$

and hence:

$$\frac{d\hat{\rho}}{dt}(t) = -i\Delta[\hat{a}^\dagger\hat{a},\hat{\rho}(t)] + \frac{\gamma}{2}\left(2\hat{a}\hat{\rho}_{(t)}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\,\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^\dagger\hat{a}\right)$$
$$+\gamma\bar{n}\left(\hat{a}\hat{\rho}_{(t)}\hat{a}^\dagger + \hat{a}^\dagger\hat{\rho}_{(t)}\hat{a} - \hat{a}^\dagger\hat{a}\,\hat{\rho}_{(t)} - \hat{\rho}_{(t)}\hat{a}\hat{a}^\dagger\right)$$

· This is the master equation in the interaction picture.
Returning to the Schrödinger picture:

$$\frac{d\hat{\rho}_{SCH}}{dt} = -\frac{i}{\hbar}[\hat{H}_S,\hat{\rho}_{SCH}] + e^{-i\hat{H}_S t/\hbar}\,\frac{d\hat{\rho}_{INT}}{dt}\,e^{i\hat{H}_S t/\hbar}$$

where $\hat{H}_S = \hbar\omega_0\hat{a}^\dagger\hat{a}$

* then (I use now $\hat{\rho} \equiv \hat{\rho}_{SCHROD.}$):

$$\frac{d\hat{\rho}}{dt} = -i\omega_0' [\hat{a}^+\hat{a}, \hat{\rho}] + \frac{\gamma}{2}(2\hat{a}\hat{\rho}\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^+\hat{a}) + \gamma\bar{n}(\hat{a}\hat{\rho}\hat{a}^+ + \hat{a}^+\hat{\rho}\hat{a} - \hat{a}^+\hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^+)$$

where $\omega_0' = \omega_0 + \Delta$

Note that the oscillator frequency is shifted when considering the coupling with the environment.

, Re-writing the master equation in a slightly different way:

$$\boxed{\frac{d\hat{\rho}}{dt} = -i\omega_0' [\hat{a}^+\hat{a}, \hat{\rho}] + \frac{\gamma}{2}(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^+\hat{a}) + \frac{\gamma}{2}\bar{n}(2\hat{a}^+\hat{\rho}\hat{a} - \hat{a}\hat{a}^+\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^+)}$$

This is the so-called Linblad form of the Master equation.

* The physical interpretation of this equation can be understood by considering the evolution of $p_n \equiv \langle n|\hat{\rho}|n\rangle \equiv$ probability to find the oscillator in its n^{th} energy eigenstate.

$$\frac{dp_n}{dt} = \underbrace{[\gamma(\bar{n}+1)(n+1)]p_{n+1}}_{\text{this comes from } 2\hat{a}\hat{\rho}\hat{a}^+} + \underbrace{(\gamma\bar{n}n)p_{n-1}}_{\text{this comes from } 2\hat{a}^+\hat{\rho}\hat{a}} - \underbrace{\gamma\bar{n}(2n+1)p_n}_{\text{this comes from } -(\hat{a}^+\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^+\hat{a}) - (\hat{a}\hat{a}^+\hat{\rho} + \hat{\rho}\hat{a}\hat{a}^+)}$$

This is an example of a so-called rate equation.

* We have a transition rate from $n+1$ into n → feeds p_n
and from $n-1$ into n → also feeds p_n
* We also go out from n (last term) → empties p_n.

(Note: the rate equation is a representation of the master equation in the Fock representation, we will see later representations using coherent states).

- let's use $\frac{d\hat{\rho}}{dt}(t)$ to evaluate the time evolution of some expected values (we use $\omega_0 \equiv \omega_0'$ from now on)

$$\frac{d\langle\hat{a}\rangle}{dt} = \frac{d}{dt}\mathrm{Tr}\{\hat{a}\hat{\rho}\} = \mathrm{Tr}\left\{\hat{a}\frac{d\hat{\rho}}{dt}\right\} =$$

$$= -i\omega_0 \mathrm{Tr}\{\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\} + \frac{\gamma}{2}\mathrm{Tr}\{2\hat{a}^2\hat{\rho}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\}$$

$$+ \gamma\bar{n}\mathrm{Tr}\{\hat{a}^2\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\}$$

CYCLIC PROPERTY

$$\overset{\downarrow}{=} -i\omega_0 \mathrm{Tr}\{(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})\hat{a}\hat{\rho}\} + \frac{\gamma}{2}\mathrm{Tr}\{(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\hat{a}\hat{\rho}\}$$

$$+ \gamma\bar{n}\mathrm{Tr}\{(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\hat{a}\hat{\rho} + \hat{a}(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})\hat{\rho}\} \overset{[\hat{a},\hat{a}^\dagger]=1}{=}$$

$$= -\left(\frac{\gamma}{2} + i\omega_0\right)\langle\hat{a}\rangle$$

- $$\frac{d\langle\hat{n}\rangle}{dt} = \frac{d}{dt}\mathrm{Tr}\{\hat{a}^\dagger\hat{a}\hat{\rho}\} = \mathrm{Tr}\left\{\hat{a}^\dagger\hat{a}\frac{d\hat{\rho}}{dt}\right\}$$

$$= -i\omega_0 \mathrm{Tr}\{\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\}$$

$$+ \frac{\gamma}{2}\mathrm{Tr}\{2\hat{a}^\dagger\hat{a}^2\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\}$$

$$+ \gamma\bar{n}\mathrm{Tr}\{\hat{a}^\dagger\hat{a}^2\hat{\rho}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\}$$

$$= \gamma\mathrm{Tr}\{\hat{a}^{\dagger 2}\hat{a}^2\hat{\rho} - (\hat{a}^\dagger\hat{a})^2\hat{\rho}\} + \gamma\bar{n}\mathrm{Tr}\left[\hat{a}^{\dagger 2}\hat{a}^2\hat{\rho} + (\hat{a}\hat{a}^\dagger)^2\hat{\rho} - (\hat{a}^\dagger\hat{a})^2\hat{\rho} - \hat{a}\hat{a}^{\dagger 2}\hat{a}\hat{\rho}\right]$$

$$= -\gamma\left[\langle\hat{n}\rangle - \bar{n}\right]$$

Hence $\frac{d}{dt}\langle\bar{n}\rangle = \gamma\bar{n} - \gamma\langle\hat{n}\rangle$

with the solution:

$$\langle\hat{n}(t)\rangle = \langle\hat{n}(0)\rangle e^{-\gamma t} + \bar{n}(1 - e^{-\gamma t})$$

- Note that for $t >> 1/\gamma$

$$\langle \hat{n}(t) \rangle \longrightarrow \bar{n}$$

i.e. after a transient time the system reaches $\bar{n}(\omega_0, T)$ i.e. the thermal-equilibrium expected values. After $t >> 1/\gamma$ all the information about the initial state is lost.

- Actually one can show (exercise) that the thermal density operator

$$\hat{\rho}_T = \frac{e^{-\hbar\omega_0 \hat{a}^+\hat{a}/K_BT}}{1-e^{-\hbar\omega_0/K_BT}}$$

is a stationary solution of the master equation, i.e. $\frac{d\hat{\rho}_T}{dt} = 0$.

This means that the harmonic oscillator in contact with a thermal reservoir becomes after a sufficiently large time $>> 1/\gamma$ in thermal equilibrium with the reservoir.

(remember that this is indeed the idea behind the standard way of introducing the canonical ensemble in statistical mechanics!)

- As a final observation, note that $\frac{d}{dt}\mathrm{Tr}\{\hat{\rho}(t)\} = 0$ and hence $\frac{d}{dt}\langle [\hat{a}, \hat{a}^+](t) \rangle = 1$ (exercise).

• QUANTUM REGRESSION THEOREM

* With the help of the master equation we can easily calculate the time evolution of expected values, like $\langle \hat{a}(t) \rangle$.
· But what about two-time correlations like $G^{(1)}(t, t+\tau) \propto \langle \hat{a}^{+}(t)\, \hat{a}(t+\tau) \rangle$?

Recall that we consider a system S in contact with a reservoir R.
The density operator for the whole $S+R$ is $\hat{W}(t)$, whereas $\hat{\rho}(t) = \mathrm{Tr}_R\{\hat{W}(t)\}$ fulfills the master equation, which we symbolically denote:

$$\frac{d}{dt}\hat{\rho} = \mathcal{L}\hat{\rho}$$

where $\mathcal{L}$ is the so-called generalized Liouvillian or superoperator.
e.g. for the damped oscillator

$$\mathcal{L}\hat{O} \equiv -i\omega_0 [\hat{a}^{+}\hat{a}, \hat{O}] + \frac{\gamma}{2}(2\hat{a}\hat{O}\hat{a}^{+} - \hat{a}^{+}\hat{a}\hat{O} - \hat{O}\hat{a}^{+}\hat{a}) + \gamma\bar{n}(\hat{a}\hat{O}\hat{a}^{+} + \hat{a}^{+}\hat{O}\hat{a} - \hat{a}^{+}\hat{a}\hat{O} - \hat{O}\hat{a}\hat{a}^{+})$$

· Let's consider now two system operators in the Heisenberg picture $\hat{O}_1(t), \hat{O}_2(t)$.
In the Heisenberg picture we can "easily" calculate the ~~average~~ correlation

$$\langle \hat{O}_1(t)\, \hat{O}_2(t') \rangle = \mathrm{Tr}_{S+R}\left[\hat{W}(0)\, \hat{O}_1(t)\, \hat{O}_2(t')\right]$$

where
$$\left.\begin{aligned} \frac{d}{dt}\hat{O}_1 &= \frac{-i}{\hbar}[\hat{O}_1, \hat{H}] \\ \frac{d}{dt}\hat{O}_2 &= \frac{-i}{\hbar}[\hat{O}_2, \hat{H}] \end{aligned}\right\} \text{Heisenberg equations.}$$

These equations have a formal solution:

$$\hat{O}_1(t) = e^{i\hat{H}t/\hbar}\, \hat{O}_1(0)\, e^{-i\hat{H}t/\hbar}$$
$$\hat{O}_2(t) = e^{i\hat{H}t/\hbar}\, \hat{O}_2(0)\, e^{-i\hat{H}t/\hbar}$$

$$\hat{W}(0) = e^{i\hat{H}t/\hbar}\,\hat{W}(t)\,e^{-i\hat{H}t/\hbar} \qquad \left(\text{since } \frac{dW}{dt} = \frac{-i}{\hbar}[H,\hat{W}]\right)$$

Then:

$$\langle \hat{O}_1(t)\hat{O}_2(t')\rangle = \mathrm{Tr}_{S+R}\Big[e^{\frac{i}{\hbar}\hat{H}t}\hat{W}(t)\underbrace{e^{-\frac{i}{\hbar}\hat{H}t}e^{\frac{i}{\hbar}\hat{H}t}}_{1}\hat{O}(0)e^{i\frac{\hat{H}}{\hbar}(t'-t)}\hat{O}_2(0)e^{-\frac{i}{\hbar}\hat{H}t'}\Big]$$

cyclic property

$$\stackrel{\downarrow}{=} \mathrm{Tr}_{S+R}\Big[\hat{O}_2(0)\,e^{-\frac{i}{\hbar}\hat{H}(t'-t)}\hat{W}(t)\,\hat{O}_1(0)\,e^{i\frac{\hat{H}}{\hbar}(t'-t)}\Big]$$

$$= \mathrm{Tr}_S\Big[\hat{O}_2(0)\,\mathrm{Tr}_R\Big[e^{\frac{-i\hat{H}(t'-t)}{\hbar}}\hat{W}(t)\hat{O}_1(0)e^{i\frac{\hat{H}}{\hbar}(t'-t)}\Big]\Big]$$

let $t' \geq t$; $\tau \equiv t'-t$

then: $\hat{W}_{O_1}(\tau) \equiv e^{-\frac{i}{\hbar}\hat{H}\tau}\,\hat{W}(t)\,\hat{O}_1(0)\,e^{\frac{i}{\hbar}\hat{H}\tau}$

This satisfies clearly

$$\frac{d\hat{W}_{O_1}(\tau)}{d\tau} = \frac{-i}{\hbar}[\hat{H},\hat{W}_{O_1}] \qquad \left(\text{since it evolves similarly as } \hat{W}(t) \text{ itself.}\right)$$

where $\hat{W}_{O_1}(\tau=0) = \hat{W}(t)\hat{O}_1(0)$

* let $\hat{\rho}_{O_1}(\tau) = \mathrm{Tr}_R[\hat{W}_{O_1}(\tau)]$ ⟵ which is a <u>reduced operator</u> (similar as $\hat{\rho}(t)$ before)

Then: $\hat{\rho}_{O_1}(\tau=0) = \mathrm{Tr}_R[\hat{W}(t)\hat{O}_1(0)] \underset{\substack{\uparrow\\ \text{since } \hat{O}_1 \text{ is}\\ \text{a system operator}}}{=} \mathrm{Tr}_R(\hat{W}(t)]\,\hat{O}_1(0) = \hat{\rho}(t)\hat{O}_1(0)$

At any time $\hat{W}(t) = \hat{\rho}(t)\otimes R_0$

then $\hat{W}_{O_1} = R_0\otimes\hat{\rho}(t)\hat{O}_1(0) = R_0\,\hat{\rho}_{O_1}(\tau=0)$

* Then $\hat{\rho}_{O_1}$ actually acts very similarly as $\hat{\rho}(t)$.

* After writing the master equation (in Born-Markov approximation) for $\hat{\rho}_{O_1}$ we get

$$\frac{d\rho_{O_1}}{dt} = \mathcal{L}\hat{\rho}_{O_1} \longrightarrow \hat{\rho}_{O_1}(\tau) = e^{\mathcal{L}\tau}\hat{\rho}_{O_1}(0) = e^{\mathcal{L}\tau}\left(\hat{\rho}(t)\,\hat{O}_1(0)\right)$$

Hence

$$\langle \hat{O}_1(t)\,\hat{O}_2(t+\tau)\rangle = \mathrm{Tr}_S\left\{\hat{O}_2(0)\, e^{\mathcal{L}\tau}\left(\hat{\rho}(t)\,\hat{O}_1(0)\right)\right\}$$

<u>Note</u>: if $t'<t$, $\tau > 0$:

$$\langle \hat{O}_1(t+\tau)\,\hat{O}_2(t)\rangle = \mathrm{Tr}_S\left\{\hat{O}_1(0)\, e^{\mathcal{L}\tau}\left(\hat{O}_2(0)\,\hat{\rho}(t)\right)\right\}$$

* These expressions are the so-called <u>quantum regression theorem</u> for 2-time averages.

* These expressions look however complicated. We will see now that actually it is quite simple!

* Let's assume that there exists a complete set of system operators (This can be proved in any case) $\hat{A}_\mu$, $\mu = 1,2,\dots$, in the sense that

$$\mathrm{Tr}_S\left[\hat{A}_\mu(\mathcal{L}\hat{O})\right] = \sum_\lambda M_{\mu\lambda}\,\mathrm{Tr}_S\left[\hat{A}_\lambda\hat{O}\right]$$

where $\hat{O}$ is an arbitrary operator.
and $M_{\mu\lambda}$ are constants.

* From this it follows that

$$\frac{d}{dt}\langle\hat{A}_\mu\rangle = \mathrm{Tr}_S\left(\hat{A}_\mu\frac{d\hat{\rho}}{dt}\right) = \mathrm{Tr}_S\left\{\hat{A}_\mu(\mathcal{L}\hat{\rho})\right\} =$$
$$= \sum_\lambda M_{\mu\lambda}\,\mathrm{Tr}_S\left[\hat{A}_\lambda\hat{\rho}\right] = \sum_\lambda M_{\mu\lambda}\langle\hat{A}_\lambda\rangle$$

Thus the expectation values obey a coupled set of linear equations (typically this builds an infinite set of equations (this is so the case for example in the damped H.O. if one takes $\hat{A}_1 = \hat{a}$) but this is not a concern here).

• Now, using the quantum regression theorem

$$\frac{d}{d\tau}\langle \hat{O}_1(t)\,\hat{A}_\mu(t+\tau)\rangle = \frac{d}{d\tau}\,\mathrm{Tr}_S\left\{\hat{A}_\mu(0)\,e^{\mathcal{L}\tau}\left[\hat{\rho}(t)\hat{O}_1(0)\right]\right\}$$

$$= \mathrm{Tr}_S\left\{\hat{A}_\mu(0)\,\mathcal{L}\left[e^{\mathcal{L}\tau}\left[\hat{\rho}(t)\hat{O}_1(0)\right]\right]\right\}$$

$$= \sum_\lambda M_{\mu\lambda}\,\mathrm{Tr}_S\left\{\hat{A}_\lambda(0)\,e^{\mathcal{L}\tau}\left[\hat{\rho}(t)\,\hat{O}_1(0)\right]\right\}$$

$$= \sum_\lambda M_{\mu\lambda}\,\langle \hat{O}_1(t)\,\hat{A}_\lambda(t+\tau)\rangle$$

Hence for each operator $\hat{O}_1$, the correlations $\langle \hat{O}_1(t)\,\hat{A}_\mu(t+\tau)\rangle$ with $\tau \geq 0$, satisfies exactly the same equations as

$\langle \hat{A}_\mu(t+\tau)\rangle$

Similarly one can prove the same for $\langle \hat{A}(t+\tau)\,\hat{O}_2(t)\rangle$ for $\tau<0$.

* So, at the end of the day the quantum regression theorem can be very easily formulated:

You know what happens for $\langle \hat{A}_\mu(t+\tau)\rangle$? If yes, then you know $\langle \hat{O}_1(t)\,\hat{A}_\mu(t+\tau)\rangle$.

* Let's see an example, namely the damped harmonic oscillator that we have seen before.

We know that

$$\frac{d}{dt}\langle \hat{a}(t)\rangle = -\left(\frac{\gamma}{2}+i\omega_0\right)\langle \hat{a}\rangle$$

Let $\hat{A}_1 = \hat{a}$, and $\hat{O}_1 = \hat{a}^+$. Then, applying the quantum regression theorem we arrive to:

$$\frac{d}{d\tau}\langle \hat{a}^+(t)\,\hat{a}(t+\tau)\rangle = -\left(\frac{\gamma}{2}+i\omega_0\right)\langle \hat{a}^+(t)\,\hat{a}(t+\tau)\rangle$$

Hence:

$$\langle \hat{a}^+(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}^+(t)\hat{a}(t)\rangle\, e^{-\left(\frac{\gamma}{2}+i\omega_0\right)\tau} =$$ remember the previous calculation

$$= \left[\langle \hat{n}(0)\rangle e^{-\gamma t} + \bar{n}\left(1-e^{-\gamma t}\right)\right] e^{-\left(\frac{\gamma}{2}+i\omega_0\right)\tau}$$

Note that for large t (as discussed before)

$$\langle \hat{a}^+(t)\,\hat{a}(t+\tau)\rangle \xrightarrow[t\to\infty]{} \langle \hat{a}^+(0)\hat{a}(\tau)\rangle_{\text{STATIONARY STATE}}$$

where $\langle \hat{a}^+(0)\hat{a}(\tau)\rangle = \bar{n}\, e^{-\left(\frac{\gamma}{2}+i\omega_0\right)\tau}$

Then $G^{(1)}(\tau) \propto \bar{n}\, e^{-\left(\frac{\gamma}{2}+i\omega_0\right)\tau}$

As expected $|G^{1}(\tau)| \propto \bar{n}\, e^{-\frac{\gamma}{2}\tau} \xrightarrow[\tau\to\infty]{} 0$

---

* let's see now the <u>evolution of $G^{(2)}$</u>.

Remember that $\frac{d}{dt}\langle \hat{n}\rangle = -\gamma\left[\langle \hat{n}\rangle - \bar{n}\right]$

. Remember that due to the quantum regression theorem

$$\frac{d}{d\tau}\langle A_\mu^+(t+\tau)\hat{O}_2(t)\rangle = \sum_\lambda M_{\mu\lambda}\langle A_\lambda(t+\tau)\,\hat{O}_2(t)\rangle$$

$$\frac{d}{d\tau}\langle \hat{O}_1(t)\,\hat{A}_\mu(t+\tau)\rangle = \sum_\lambda M_{\mu\lambda}\langle \hat{O}_1(t)\hat{A}_\lambda(t+\tau)\rangle$$

then: $\frac{d}{d\tau}\langle \hat{O}_1(t)\,\hat{A}_\mu(t+\tau)\,\hat{O}_2(t)\rangle = \sum_\lambda M_{\mu\lambda}\langle \hat{O}_1(t)\,\hat{A}_\lambda(t+\tau)\,\hat{O}_2(t)\rangle$

* This is very useful to calculate G_2.

First we do a useful trick:

we just write $\frac{d\langle\hat{n}\rangle}{dt} = -\gamma[\langle\hat{n}\rangle - \bar{n}]$ in a ~~matrix~~ matrix way

$$\frac{d}{dt}\begin{pmatrix}\langle\hat{n}\rangle \\ \bar{n}\end{pmatrix} = \begin{pmatrix}-\gamma & \gamma \\ 0 & 0\end{pmatrix}\begin{pmatrix}\langle\hat{n}\rangle \\ \bar{n}\end{pmatrix}$$

Then now $\hat{A}_1 = \hat{n} = \hat{a}^{+}\hat{a}$

$\hat{A}_2 = \bar{n} \equiv$ constant

let $\hat{B}_1 = \hat{a}^{+}$ and $\hat{B}_2 = \hat{a}$, then:

$$\frac{d}{d\tau}\begin{pmatrix}\langle\hat{a}^{+}(t)\,\hat{n}(t+\tau)\,\hat{a}(t)\rangle \\ \bar{n}\langle\hat{a}^{+}(t)\,\hat{a}(t)\rangle\end{pmatrix} = \begin{pmatrix}-\gamma & \gamma \\ 0 & 0\end{pmatrix}\begin{pmatrix}\langle\hat{a}^{+}(t)\,\hat{n}(t+\tau)\,\hat{a}(t)\rangle \\ \bar{n}\langle\hat{n}(t)\rangle\end{pmatrix}$$

then:

$$\frac{d}{d\tau}\langle\hat{a}^{+}(t)\,\hat{n}(t+\tau)\,\hat{a}(t)\rangle = -\gamma\langle\hat{a}^{+}(t)\,\hat{n}(t+\tau)\,\hat{a}(t)\rangle + \gamma\bar{n}\langle\hat{n}(t)\rangle$$

Then using the expression we got before for $\langle\hat{n}(t)\rangle$:

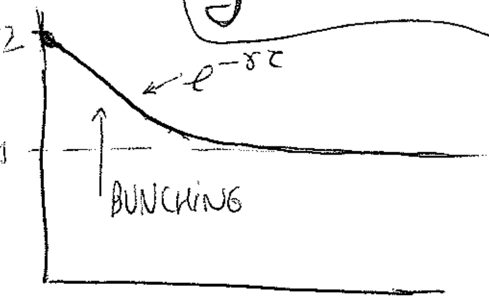
$$\langle\hat{a}^{+}(t)\,\hat{n}(t+\tau)\,\hat{a}(t)\rangle = \langle\hat{a}^{+}(t)\,\hat{n}(t)\,\hat{a}(t)\rangle e^{-\gamma\tau} + \bar{n}\langle\hat{n}(t)\rangle(1-e^{-\gamma\tau})$$

In the long time (t) limit the stationary limit is reached

$$\langle\hat{a}^{+}(0)\,\hat{n}(\tau)\,\hat{a}(0)\rangle = \underbrace{\langle\hat{a}^{+}(0)\,\hat{a}^{+}(0)\,\hat{a}(0)\,\hat{a}(0)\rangle}_{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle} e^{-\gamma\tau} + \bar{n}\langle\hat{n}(t)\rangle(1-e^{-\gamma\tau})$$

For a chaotic (thermal) state $\langle\hat{n}^2\rangle = 2\bar{n}^2 + \bar{n} \rightarrow \langle\hat{n}^2\rangle - \langle\hat{n}\rangle = 2\bar{n}^2$

Hence: $$\boxed{g^{(2)}(\tau) = \frac{\langle\hat{a}^{+}(0)\,\hat{a}^{+}(\tau)\,\hat{a}(\tau)\,\hat{a}(0)\rangle}{\bar{n}^2} = 1 + e^{-\gamma\tau}}$$



* So we recover 2 things we knew already
  * Bunching for small τ
  * $g^{(2)}(\tau)$ goes to 1 for large τ.

Correlation time $t_c = 1/\gamma$

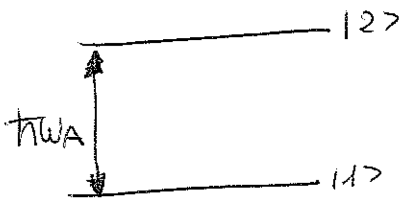
* THE SPONTANEOUS EMISSION IN A TWO-LEVEL ATOM

* Remember that we have already analyze how light interacts with a two level atom. After doing several reasonable approximations (as dipole approximation and rotating-wave approximation) we showed that the light atom interaction was provided in the pseudo-spin notation by the Jaynes-Cummings Hamiltonian:

$$\hat{H} = \frac{\hbar\omega_A}{2}\hat{\sigma}_z + \hbar\omega\,\hat{a}^+\hat{a} + \hbar g(\hat{\sigma}^-\hat{a}^+ + \hat{\sigma}^+\hat{a})$$

where $\hat{\sigma}_z \equiv \hat{b}_2^+\hat{b}_1 - \hat{b}_1^+\hat{b}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

where



$\hat{\sigma}^+ = 2\hat{b}_2^+\hat{b}_1 \equiv 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\hat{\sigma}^- = 2\hat{b}_1^+\hat{b}_2 \equiv 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

and $\hat{a}^+, \hat{a}$ was the EM mode consider

g = coupling constant

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* In the following we consider an atom that is radiatively damped by its interaction not with just one mode, but with many modes of the EM field in thermal equilibrium at some temperature T (which act as a reservoir of harmonic oscillators).

Then, now:

- SYSTEM: two-level atom $\longrightarrow \hat{H}_S = \frac{\hbar\omega_A}{2}\hat{\sigma}_z$
- RESERVOIR: Modes of the EM in thermal equilibrium $\Big\} \longrightarrow \hat{H}_R = \sum_{k,\lambda}\hbar\omega_k\,\hat{r}_{k\lambda}^+\hat{r}_{k\lambda}$

* COUPLING: (in RWA) $\hat{H}_{SR} = \sum_{k,\lambda}\hbar(k_{k\lambda}^*\,\hat{r}_{k\lambda}^+\hat{\sigma}_- + k_{k\lambda}\hat{r}_{k\lambda}\hat{\sigma}_+)$

where $k_{k\lambda} = -i\,e^{i\vec{k}\vec{r}_A}\sqrt{\frac{\omega_k}{2\hbar\epsilon_0 V}}\,\hat{e}_{k\lambda}\cdot\vec{d}_{21}$

$d_{21} \equiv e\langle 1|\hat{\vec{r}}|2\rangle$

* Remember the general coupling Hamiltonian (see p. 76):

$$\hat{H}_{SR} = \hbar \sum_i \hat{S}_i \hat{\Gamma}_i$$

Now: $\hat{S}_1 = \hat{\sigma}_-$; $\hat{S}_2 = \hat{\sigma}_+$

$$\hat{\Gamma}_1 = \hat{\Gamma}^+ \equiv \sum_{k\lambda} \kappa^*_{k\lambda} \hat{r}^+_{k\lambda} \quad ; \quad \hat{\Gamma}_2 = \Gamma \equiv \sum_{k,\lambda} \hat{r}_{k\lambda}$$

. In the interaction picture with $\hat{H}_S + \hat{H}_R$:

$$\hat{\Gamma}_1(t) = \sum_{k,\lambda} \kappa^*_{k\lambda} \hat{r}^+_{k\lambda} e^{i\omega_k t}$$

$$\hat{\Gamma}_2(t) = \sum_{k,\lambda} \kappa_{k\lambda} \hat{r}_{k\lambda} e^{-i\omega_k t}$$

$$\hat{S}_1(t) = e^{i\omega_A \frac{\hat{\sigma}_z}{2} t} \hat{\sigma}_- e^{-i\omega_A \frac{\hat{\sigma}_z}{2}} = \hat{\sigma}_- e^{-i\omega_A t}$$

$$\hat{S}_2(t) = \hat{\sigma}_+ e^{i\omega_A t}$$

* The derivation of the master equation for a two-level atom follows in complete analogy to the derivation of the master equation for the harmonic oscillator, except that

i) The summation over reservoir oscillators now involves a summation over polarizations (λ) and wavevectors $\vec{k}$.

ii) The commutation relations for the Pauli matrices is obviously different than for $\hat{a}$ and $\hat{a}^+$. (remember that $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$
$[\sigma_\pm, \hat{\sigma}_z] = \mp 2\hat{\sigma}_\pm$)

. One gets:

$$\frac{d\hat{\rho}}{dt} = \left[\frac{\gamma}{2}(\bar{n}+1) + i(\Delta' + \Delta)\right]\left[\hat{\sigma}_- \hat{\rho}\, \hat{\sigma}_+ - \hat{\sigma}_+ \hat{\sigma}_- \hat{\rho}\right]$$

$$+ \left(\frac{\gamma}{2}\bar{n} + i\Delta'\right)\left(\hat{\sigma}_+ \hat{\rho}\, \hat{\sigma}_- - \hat{\rho}\, \hat{\sigma}_- \hat{\sigma}_+\right) + h.c.$$

where:

$$\gamma \equiv 2\pi \sum_\lambda \int d^3k\, g(\vec{k})\, |k(\vec{k},\lambda)|^2\, \delta(kc-\omega_A)$$

$$\Delta \equiv \sum_\lambda P \int d^3k\, g(\vec{k})\, \frac{|k(k,\lambda)|^2}{\omega_A - kc}$$

$$\Delta' \equiv \sum_\lambda P \int d^3k\, g(\vec{k})\, \frac{|k(\vec{k},\lambda)|^2}{\omega_A - kc}\, \bar{n}(kc,T)$$

Then:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{2}(\Delta + 2\Delta')[\hat{\sigma}_z,\hat{\rho}] + \frac{\gamma}{2}(\bar{n}+1)\left[2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right) + \frac{\gamma}{2}\bar{n}\left(2\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \hat{\sigma}_-\hat{\sigma}_+\hat{\rho} - \hat{\rho}\hat{\sigma}_-\hat{\sigma}_+\right)$$

This is the interaction picture. We will go now back (as we did for the damped oscillator) into the Schrödinger picture:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{2}\omega_A'[\hat{\sigma}_z,\hat{\rho}] + \frac{\gamma}{2}(\bar{n}+1)\left[2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right] + \frac{\gamma}{2}\bar{n}\left[2\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \hat{\sigma}_-\hat{\sigma}_+\hat{\rho} - \hat{\rho}\hat{\sigma}_-\hat{\sigma}_+\right]$$

with $\omega_A' = \omega_A + \Delta + 2\Delta'$

* As for the damped harmonic oscillator we can understand the different terms in the master equation in a "microscopic" way, by having a look to $P_n = \langle n|\hat{\rho}|n\rangle$, where now $n = 1,2$:

This comes from $2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+$

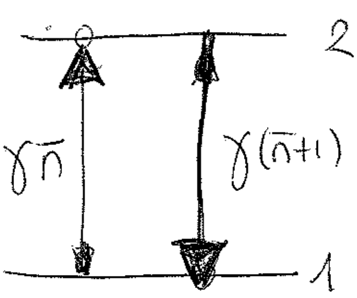
$$\frac{dP_1}{dt} = +\gamma(\bar{n}+1)P_2 - \gamma\bar{n}P_1$$

rest

$$\frac{dP_2}{dt} = -\gamma(\bar{n}+1)P_2 + \gamma\bar{n}P_1$$

rest

from $2\hat{\sigma}_+\hat{\rho}\hat{\sigma}_-$



(94)

(Note: as it should be $\frac{d}{dt}(\rho_1+\rho_2)=0$, since $\rho_1+\rho_2=1$)

* So the rate equation tells us that the transition $|2\rangle \to |1\rangle$ has a rate $\gamma(\bar{n}+1)$ and $|1\rangle \to |2\rangle$ a rate $\gamma\bar{n}$.

- Note that $|2\rangle \to |1\rangle$ has a rate even for $\bar{n}=0$.

Hence:

$|2\rangle \to |1\rangle$: $\begin{cases} \gamma\bar{n} \longrightarrow \text{stimulated transition induced by thermal photons} \\ \gamma \longrightarrow \text{spontaneous emission (independent of } \bar{n}) \end{cases}$

$|1\rangle \to |2\rangle$: $\gamma\bar{n} \longrightarrow$ absorptive transitions which take thermal photons from the EM field

- Notice that the lamb shift $\omega_A' - \omega_A$ includes $2\Delta'$ which depends on temperature.

- Hence γ can be interpreted as <u>the spontaneous emission rate</u>

Let's see its form.

We will perform the integral $\int d^3k$ in spherical coordinates.

The density of states for each polarization λ is

$$g(\bar{k})\, d^3k \Longrightarrow \frac{\omega^2 V}{8\pi^3 c^3}\, d\omega \sin\theta\, d\theta\, d\phi$$

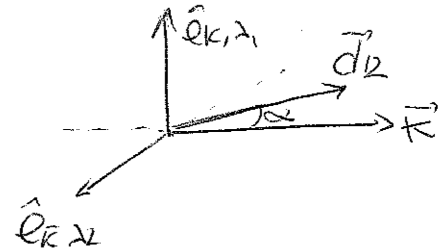
[Note: $\sum_{k} \longrightarrow \frac{V}{(2\pi)^3}\int d^3k = \frac{V}{(2\pi)^3}\int k^2 dk \sin\theta\, d\theta\, d\phi \overset{\omega = kc}{=} \int\left(\frac{\omega^2 V}{8\pi^3 c^3}\right) d\omega \sin\theta\, d\theta\, d\phi$]

• Hence:

$$\gamma = 2\pi \sum_\lambda \int_0^\infty d\omega \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \; \frac{\omega^2 V}{8\pi^3 c^3} \; \frac{\omega}{2\hbar\epsilon_0 V} \left(\hat{e}_{k\lambda}\cdot\vec{d}_{12}\right)^2 \delta(\omega-\omega_A)$$

$$= \frac{\omega_A^3}{8\pi^2\epsilon_0\hbar c^3} \sum_\lambda \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \; \left(\hat{e}_{k\lambda}\cdot\vec{d}_{12}\right)^2$$

* We have 2 independent orthogonal polarizations. Let's choose one such that $\hat{e}_{k\lambda_1}\cdot\vec{d}_{12}=0$ (remember that $\hat{e}_{k\lambda}\cdot\vec{k}=0$)



then $(\hat{e}_{k\lambda_2}\cdot\vec{d}_{12})^2 = d_{12}^2(1-\cos^2\alpha)$

$= d_{12}^2\left(1-(\hat{d}_{12}\cdot\hat{k})^2\right]$

• Let the k_z axis the direction of $\vec{d}_{12} \longrightarrow \hat{d}_{12}\cdot\hat{k}=\cos\theta$

and hence:

$$\gamma = \frac{\omega_A^3}{8\pi^2\epsilon_0\hbar c^3} \underbrace{\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \; (1-\cos^2\theta)}_{8\pi/3} d_{12}^2 \Rightarrow$$

$$\Rightarrow \boxed{\gamma = \frac{1}{4\pi\epsilon_0}\,\frac{4\omega_A^3 d_{12}^2}{3\hbar c^3}}$$ (this is the so-called Einstein's A coefficient)

See that the spontaneous emission rate depends

• on the dipole like d_{12}^2

• on the transition frequency like ω_A^3

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* In the following we will calculate ~~some~~ averages of interest by means of the master equation and the quantum regression theorem.