* THE FOKKER-PLANCK EQUATION

· In our discussion of stochastic methods, we have introduced up to now mainly 2 important techniques

- The Master equation
- The Quantum Regression theorem

with these techniques we have solved some rather important problems as the damped harmonic oscillator, the spontaneous emission and the resonance fluorescence.

* I would not like to finish the discussion about stochastic methods without mentioning other very powerful technique, which merges what we have learned about the Master equation and about the representations (recall P, Q and W). This technique is the so-called Fokker-Planck equation. (FPE) Unfortunately we don't have here time to go through all the applications of this important equation, but we will at least derive the FPE for P, Q and W, and use it for a particular example, namely the damped harmonic oscillator we found already before.

. We are going to see that the master equation for a damped harmonic oscillator (which is an equation for the density operator, and hence hard to solve) can be reduced to a differential equation for the P representation (we will see later that it can be also done with Q or W).

* Fokker–Planck equation for the damped harmonic oscillator

* Remember our discussion about the damped harmonic oscillator.
In that discussion we had an harmonic oscillator ($\hbar\omega_0 \hat{a}^+\hat{a}$)
in contact with a thermal reservoir (characterized by $\bar{n} = \bar{n}(T)$).
The evolution of the reduced density operator $\hat{\rho}$ was given
by the corresponding master equation (see p. 82)

$$\frac{d\hat{\rho}}{dt} = -i\omega_0 [\hat{a}^+\hat{a}, \hat{\rho}] + \frac{\gamma}{2}[2\hat{a}\hat{\rho}\hat{a}^+ - \hat{a}^+\hat{a}\rho - \rho\hat{a}^+\hat{a}]$$
$$+ \gamma\bar{n}[\hat{a}\hat{\rho}\hat{a}^+ + \hat{a}^+\hat{\rho}\hat{a} - \hat{a}^+\hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^+]$$

* Remember (p. 58) our definition of the P-representation

$$\hat{\rho} = \int d^2\alpha\, |\alpha\rangle\langle\alpha|\, P(\alpha)$$

* Our goal now is to convert the operator Master equation into
an equation of motion for P.
(We assume that such P exists, which as we already discussed
is not always the case, remember the squeezed states)

* So we substitute into the Master equation: superoperator (remember p. 85)

$$\int d^2\alpha\, |\alpha\rangle\langle\alpha| \frac{\partial}{\partial t} P(\alpha,t) = \int d^2\alpha\, P(\alpha)\, \mathcal{L}\,[|\alpha\rangle\langle\alpha|]$$
$$= \int d^2\alpha\, P(\alpha) \Big[-i\omega_0 (\hat{a}^+\hat{a}|\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|\hat{a}^+\hat{a})$$
$$+ \frac{\gamma}{2}[2\hat{a}|\alpha\rangle\langle\alpha|\hat{a}^+ - \hat{a}^+\hat{a}|\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|\hat{a}^+\hat{a}]$$
$$+ \gamma\bar{n}[\hat{a}|\alpha\rangle\langle\alpha|\hat{a}^+ + \hat{a}^+|\alpha\rangle\langle\alpha|\hat{a} - \hat{a}^+\hat{a}|\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|\hat{a}\hat{a}^+] \Big]$$

* Let's recall now some properties of the coherent states $|\alpha\rangle$

- By definition $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$; $\langle\alpha|\hat{a}^+ = \alpha^*\langle\alpha|$
- Also $|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^+}|0\rangle$ (this can be obtain from an expression in p. 7)
- Hence $|\alpha\rangle\langle\alpha| = e^{-|\alpha|^2} e^{\alpha\hat{a}^+}|0\rangle\langle 0| e^{\alpha^*\hat{a}}$

and therefore:

$$\frac{\partial}{\partial\alpha^*}|\alpha\rangle\langle\alpha| = |\alpha\rangle\langle\alpha|[\hat{a}-\alpha]$$

- Thus: $\hat{a}|\alpha\rangle\langle\alpha| = \alpha|\alpha\rangle\langle\alpha| \longrightarrow |\alpha\rangle\langle\alpha|\hat{a}^+ = \alpha^*|\alpha\rangle\langle\alpha|$

$$|\alpha\rangle\langle\alpha|\hat{a} = \left(\frac{\partial}{\partial\alpha^*}+\alpha\right)|\alpha\rangle\langle\alpha| \longrightarrow \hat{a}^+|\alpha\rangle\langle\alpha| = \left(\frac{\partial}{\partial\alpha}+\alpha^*\right)|\alpha\rangle\langle\alpha|$$

- Using these properties we can easily evaluate the different terms coming from the master equation:

$$\hat{a}^+\hat{a}|\alpha\rangle\langle\alpha| = \alpha\left(\frac{\partial}{\partial\alpha}+\alpha^*\right)|\alpha\rangle\langle\alpha|$$

$$|\alpha\rangle\langle\alpha|\hat{a}^+\hat{a} = \alpha^*\left(\frac{\partial}{\partial\alpha^*}+\alpha\right)|\alpha\rangle\langle\alpha|$$

$$\hat{a}|\alpha\rangle\langle\alpha|\hat{a}^+ = |\alpha|^2|\alpha\rangle\langle\alpha|$$

$$|\alpha\rangle\langle\alpha|\hat{a}\hat{a}^+ = \left(\frac{\partial}{\partial\alpha^*}+\alpha\right)|\alpha\rangle\langle\alpha|\hat{a}^+ = \left(\frac{\partial}{\partial\alpha^*}+\alpha\right)\alpha^*|\alpha\rangle\langle\alpha|$$

$$\hat{a}^+|\alpha\rangle\langle\alpha|\hat{a} = \left(\frac{\partial}{\partial\alpha}+\alpha^*\right)|\alpha\rangle\langle\alpha|\hat{a} = \left(\frac{\partial}{\partial\alpha}+\alpha^*\right)\left(\frac{\partial}{\partial\alpha^*}+\alpha\right)|\alpha\rangle\langle\alpha|$$

* Substituting this into the master equation we get:

$$\int d^2\alpha\, |\alpha\rangle\langle\alpha|\frac{\partial}{\partial t}P(\alpha,t) = \int d^2\alpha\, P(\alpha,t)\left[-\left(\frac{\gamma}{2}+i\omega_0\right)\alpha\frac{\partial}{\partial\alpha} - \left(\frac{\gamma}{2}-i\omega_0\right)\alpha^*\frac{\partial}{\partial\alpha^*} + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]|\alpha\rangle\langle\alpha|$$

* We will now integrate by parts, assuming that $P(\alpha,t)$ vanishes sufficiently rapidly as $|\alpha|\to\infty$ to drop the boundary terms.

(<u>Note</u>: $\int d^2\alpha\, f(\alpha)\frac{\partial}{\partial\alpha}g(\alpha) = \text{BOUNDARY TERMS} - \int d^2\alpha\, g(\alpha)\frac{\partial}{\partial\alpha}f(\alpha)$)

Hence:

$$\int d^2\alpha\, |\alpha\rangle\langle\alpha| \frac{\partial}{\partial t}P(\alpha,t) = \int d^2\alpha\, |\alpha\rangle\langle\alpha| \left[\left(\frac{\gamma}{2}+i\omega_0\right)\frac{\partial}{\partial\alpha}\alpha + \left(\frac{\gamma}{2}-i\omega_0\right)\frac{\partial}{\partial\alpha^*}\alpha^* + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]P(\alpha,t)$$

· A sufficient condition to satisfy this equation is that

$$\boxed{\frac{\partial}{\partial t}P(\alpha,t) = \left[\left(\frac{\gamma}{2}+i\omega_0\right)\frac{\partial}{\partial\alpha}\alpha + \left(\frac{\gamma}{2}-i\omega_0\right)\frac{\partial}{\partial\alpha^*}\alpha^* + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]P(\alpha,t)}$$

This is <u>the Fokker-Planck equation for the damped harmonic oscillator in the P-representation</u>

(<u>Note</u>: just to avoid confusions. We wrote $\left[\frac{\partial}{\partial\alpha}\alpha\right]P$, this means of course $\frac{\partial}{\partial\alpha}(\alpha P)$)

* The Fokker-Planck equation was introduced (as its name suggests) by Fokker and Planck in ~~the~~ 1910's to describe Brownian motion, and it is a key tool for the understanding of classical stochastic processes. It has many interesting properties and applications, but we have no time here to have a look to them in detail. (Some general considerations are in p. 121' and 121'')

* We are going to solve now this equation for the particular case of a damped harmonic oscillator which is initially in a coherent state $|\alpha_0\rangle$.

* THE FOKKER-PLANCK EQUATION

- A general Fokker-Planck equation in n variables may be written in the form:

$$\frac{\partial}{\partial t}P(\vec{x}) = -\left[\frac{\partial}{\partial x_j}A_j(\vec{x}) + \frac{1}{2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}D_{ij}(\vec{x})\right]P(\vec{x})$$ (we employ Einstein's sum convention)

- Note that in the case of the damped harmonic oscillator x_1 and x_2 are α and α^*.

- The first derivative term determines the mean or deterministic motion and is called the drift term

The 2^{nd} derivative term, provided its coefficient is positive definite, will cause a broadening or diffusion of $P(\vec{x},t)$ and is hence called the diffusion term

- $\vec{A} \equiv \{A_j\}$ is the drift vector

$\hat{D} \equiv \{D_{ij}\}$ is the diffusion matrix

- Note that the drift vector for the damped harmonic oscillator is $\left[\left(\frac{\gamma}{2}+i\omega_0\right)\alpha, \left(\frac{\gamma}{2}-i\omega_0\right)\alpha^*\right]$ whereas the diffusion matrix for the FPE in the P-representation is $\begin{pmatrix} 0 & -\gamma\bar{n} \\ \gamma\bar{n} & 0 \end{pmatrix}$.

- The different role of the 2 terms is easy to see in the equations of motion for $\langle x_k\rangle$ and $\langle x_k x_l\rangle$:

$$\frac{d}{dt}\langle x_k\rangle = \int dx\, x_k \frac{\partial}{\partial t}P = \int dx\, x_k\left\{-\frac{\partial}{\partial x_j}(A_j P) + \frac{1}{2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(D_{ij}P)\right\} \overset{\text{by parts}}{=}$$

$$= \int dx\left[A_j P\frac{\partial}{\partial x_j}x_k - \frac{1}{2}\frac{\partial}{\partial x_j}(D_{ij}P)\frac{\partial}{\partial x_i}x_k\right] =$$

$$= \int dx\, A_k P + \cancel{\frac{1}{2}\int dx \frac{\partial}{\partial x_j}(D_{kj} P)} = \langle A_k \rangle$$

Hence the motion of the averages $\langle x_k \rangle$ is just given by the average drift $\langle A_k \rangle$

. But

$$\frac{d}{dt}\langle x_k x_l \rangle = \int dx\, x_k x_l \frac{\partial P}{\partial t} =$$

$$= \int dx\, x_k x_l \left\{ -\frac{\partial}{\partial x_j}(A_j P) + \frac{1}{2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(D_{ij} P) \right\}$$

$$= \int dx \left[A_j P \frac{\partial}{\partial x_j}(x_k x_l) + \frac{1}{2}\frac{\partial}{\partial_j}(D_{ij} P)\frac{\partial}{\partial x_i}(x_k x_l) \right\}$$

$$= \int dx \left\{ A_j P \left[x_l \delta_{jk} + x_k \delta_{jl} \right] - \frac{1}{2}\frac{\partial}{\partial_j}(D_{ij} P)\left[x_l \delta_{ik} + x_k \delta_{il} \right] \right\}$$

$$= \int dx \left\{ \left[x_l A_k + x_k A_l \right] P - \frac{1}{2} x_l \frac{\partial}{\partial_j}(D_{kj} P) - \frac{1}{2} x_k \frac{\partial}{\partial_j}(D_{lj} P) \right\}$$

$$= \langle x_l A_k \rangle + \langle x_k A_l \rangle + \frac{1}{2}\int dx \left\{ D_{kj} P \frac{\partial}{\partial_j} x_l + D_{lj} P \frac{\partial}{\partial_j} x_k \right\}$$

$$= \langle x_l A_k \rangle + \langle x_k A_l \rangle + \frac{1}{2}\langle D_{kl} + D_{lk} \rangle$$

So we see that whereas A_k determines the motion of the mean amplitudes, D_{lk} enters into the equation for correlations.

E.g. for the damped harmonic oscillator

$$\frac{d}{dt}\langle \alpha \rangle = -\frac{\gamma}{2}\langle \alpha \rangle$$

$$\frac{d}{dt}\langle \alpha^* \alpha \rangle = -\gamma \langle \alpha^* \alpha \rangle + \gamma \bar{n}$$

* Remember (p. 58) that the P-representation for a coherent state $|\alpha_0\rangle$ is: $P(\alpha) = \delta^{(2)}(\alpha-\alpha_0)$ → This means that we are actually searching for the Green function associated to the FPE in the P representation.

* It's convenient to transform to a frame rotating with the frequency ω_0 of the harmonic oscillator:

$$\left.\begin{aligned} \alpha &= e^{-i\omega_0 t}\tilde{\alpha} \\ \alpha^* &= e^{i\omega_0 t}\tilde{\alpha}^* \end{aligned}\right\} \quad P(\alpha,\alpha^*,t) = \tilde{P}(\tilde{\alpha},\tilde{\alpha}^*,t)$$

Hence:

$$\frac{\partial \tilde{P}}{\partial t} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial P}{\partial \alpha^*}\frac{\partial \alpha^*}{\partial t}$$

$$= \frac{\partial P}{\partial t} - i\omega_0\left[\alpha\frac{\partial P}{\partial \alpha} - \alpha^*\frac{\partial P}{\partial \alpha^*}\right]$$

$$= \frac{\partial P}{\partial t} - i\omega_0\left[\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^*\right]P$$

Therefore:

$$\frac{\partial \tilde{P}}{\partial t} = \left[\frac{\gamma}{2}\left(\frac{\partial}{\partial \tilde{\alpha}}\tilde{\alpha} + \frac{\partial}{\partial \tilde{\alpha}^*}\tilde{\alpha}^*\right) + \gamma\bar{n}\frac{\partial^2}{\partial \tilde{\alpha}\,\partial \tilde{\alpha}^*}\right]\tilde{P}$$

Let $\tilde{\alpha} = \tilde{x} + i\tilde{y}$, then:

$$\frac{\partial \tilde{P}}{\partial t} = \left[\frac{\gamma}{2}\left(\frac{\partial}{\partial \tilde{x}}\tilde{x} + \frac{\partial}{\partial \tilde{y}}\tilde{y}\right) + \frac{\gamma\bar{n}}{4}\left(\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2}\right)\right]\tilde{P}$$

Note that this equation is separable in $\tilde{x}$ and $\tilde{y}$, hence

$$\tilde{P}(\tilde{x},\tilde{y},t) = X(\tilde{x},t)\,Y(\tilde{y},t), \quad \text{with:}$$

$$\frac{\partial X}{\partial t} = \left(\frac{\gamma}{2}\frac{\partial}{\partial \tilde{x}}\tilde{x} + \frac{\gamma\bar{n}}{4}\frac{\partial^2}{\partial \tilde{x}^2}\right)X \quad \rightarrow X(\tilde{x},t=0) = \delta(\tilde{x}-\tilde{x}_0)$$

$$\frac{\partial Y}{\partial t} = \left(\frac{\gamma}{2}\frac{\partial}{\partial \tilde{y}}\tilde{y} + \frac{\gamma\bar{n}}{4}\frac{\partial^2}{\partial \tilde{y}^2}\right)Y \quad \rightarrow Y(\tilde{y},t=0) = \delta(\tilde{y}-\tilde{y}_0)$$

* Let's Fourier-Transform:

$$U(\tilde{u},t) = \int_{-\infty}^{\infty} d\tilde{x}\, X(\tilde{x},t)\, e^{i\tilde{x}\tilde{u}}$$

Then:

$$\frac{\partial U}{\partial t} = -\left(\frac{\gamma}{2}\tilde{u}\frac{\partial}{\partial \tilde{u}} + \frac{\gamma \bar{n}}{4}\tilde{u}^2\right)U \qquad U(\tilde{u},0) = e^{i\tilde{x}_0\tilde{u}}$$

* Let $U = e^{-\frac{\bar{n}}{4}\tilde{u}^2}\phi(\tilde{u},t)$, then

$$\frac{\partial\phi}{\partial t} + \frac{\gamma}{2}\tilde{u}\frac{\partial}{\partial\tilde{u}}\phi = 0 \longrightarrow \left(\frac{\partial}{\partial t} + \frac{\gamma}{2}\tilde{u}\frac{\partial}{\partial\tilde{u}}\right)\phi = 0$$

Let $S = S(\tilde{u},t) \longrightarrow$

Let's assume that $\phi = \phi[S(\tilde{u},t)]$, such that

$$\left(\frac{\partial}{\partial t} + \frac{\gamma}{2}\tilde{u}\frac{\partial}{\partial\tilde{u}}\right)\phi = \left(\frac{\partial S}{\partial t} + \frac{\gamma}{2}\tilde{u}\frac{\partial S}{\partial\tilde{u}}\right)\frac{d\phi}{dS} = 0$$

Let $S = \tilde{u}e^{-\frac{\gamma}{2}t} \rightarrow \frac{\partial S}{\partial t} = -\frac{\gamma}{2}\tilde{u}\frac{\partial S}{\partial\tilde{u}}$

Hence $\frac{d\phi}{dS} = 0 \longrightarrow$ Hence ϕ is a fixed function of S.

Hence $\phi = \phi[\tilde{u}e^{-\frac{\gamma}{2}t}]$

and then $U = e^{-\frac{\bar{n}}{4}\tilde{u}^2}\phi[\tilde{u}e^{-\frac{\gamma}{2}t}]$

Initially: $U = e^{i\tilde{u}\tilde{x}_0} \rightarrow e^{i\tilde{u}\tilde{x}_0} = e^{-\frac{\bar{n}}{4}\tilde{u}^2}\phi(\tilde{u})$

$$\phi(\tilde{u}) = e^{i\tilde{u}\tilde{x}_0}e^{\bar{n}/4\,\tilde{u}^2} \longrightarrow \phi[\tilde{u}e^{-\frac{\gamma}{2}t}] = e^{i\tilde{x}_0\tilde{u}e^{-\frac{\gamma}{2}t}}e^{\frac{\bar{n}}{4}\tilde{u}^2e^{-\gamma t}}$$

Hence $U(\tilde{u},t) = e^{i\tilde{x}_0\tilde{u}e^{-\frac{\gamma}{2}t}}\,e^{-\frac{\bar{n}}{4}\tilde{u}^2(1-e^{-\gamma t})}$

* Taking the inverse Fourier Transform:
(Note: it's easy, because we have basically gaussian integrals)

$$\tilde{P}(\tilde{x},\tilde{y},t) = \frac{1}{\pi\bar{n}(1-e^{-\gamma t})} \exp\left[-\frac{\left[(\tilde{x}-\tilde{x}_0 e^{-\frac{\gamma}{2}t})^2 + (\tilde{y}-\tilde{y}_0 e^{-\frac{\gamma}{2}t})^2\right]}{\bar{n}(1-e^{-\gamma t})}\right]$$

or equivalently

$$\boxed{P(\alpha,\alpha^*,t) = \frac{1}{\pi\bar{n}(1-e^{-\gamma t})} \exp\left[-\frac{|\alpha-\alpha_0 e^{-\frac{\gamma}{2}t} e^{-i\omega_0 t}|^2}{\bar{n}(1-e^{-\gamma t})}\right]}$$

* Note that $P(\alpha,\alpha^*,t)$ has a gaussian form.
One can easily evaluate averages:

$$\langle \hat{a}(t)\rangle = \int d^2\alpha\ \alpha\, P(\alpha,\alpha^*,t) = \alpha_0 e^{-\frac{\gamma}{2}t} e^{-i\omega_0 t}$$

(remember that we got this already directly from the master equation in p. 83)

Also

$$\langle \hat{a}^\dagger\hat{a}\rangle(t) = \int d^2\alpha\ P(\alpha,\alpha^*,t)|\alpha|^2 = |\alpha_0|^2 e^{-\gamma t} + \bar{n}(1-e^{-\gamma t})$$

i.e. the result we already found in p. 83

* By means of the P representation obtained using the Fokker-Planck equation we can easily obtain averages of normally-ordered operator products.

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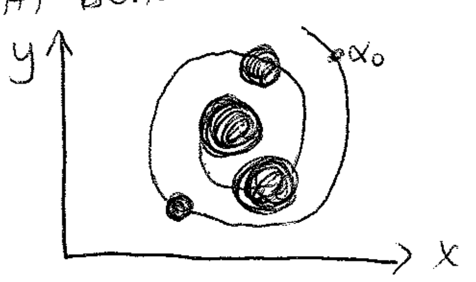
* Note: $P(\alpha,\alpha^*,t)$ behaves as a Gaussian performing a spiral motion in the xy plane and getting "fatter" in the way.



* We have seen the evolution of the P representation, when the initial state is a coherent state $|\alpha_0\rangle$. Remember that the initial state is

$$P(\alpha,\alpha^*,t=0) = \delta^{(2)}(\alpha-\alpha_0)$$

This means, as mentioned above, that

$$P(\alpha,\alpha^*,t) = P(\alpha,\alpha^*,t;\alpha_0,\alpha_0^*,0) \equiv$$ Green function associated to the Fokker-Planck equation in the P-representation.

* Remember that the Green function is crucial for the understanding of the evolution of any initial condition.

Let's consider an initial P representation $P(\lambda,\lambda^*,0)$

Then after some time t, the solution of the Fokker-Planck equation gives us:

$$\boxed{P(\alpha,\alpha^*,t) = \int d^2\lambda \; P(\alpha,\alpha^*,t;\lambda,\lambda^*,0)\; P(\lambda,\lambda^*,0)}$$

* So now, we have a general procedure to calculate the time evolution of the P-representation for any initial state, of course for all initial conditions which have a non-singular P-representation.

* Of course the Fokker-Planck equation in the P-representation is just useful if the P-representation exists. If not, then we should try with other type of representations, as Q or W.

* In order to find the corresponding form for the Q and W representations, let's first derive the Fokker-Planck equation for the normally-ordered characteristic function

$$\chi_N(\eta,\eta^*) = \mathrm{Tr}\{\hat{\rho}\, e^{\eta \hat{a}^+} e^{-\eta^* \hat{a}}\}$$

Remember that (see p. 61)

$$P(\alpha,\alpha^*) = \int \frac{d^2\eta}{\pi^2} \chi_N(\eta,\eta^*)\, e^{\alpha\eta^*} e^{-\alpha^*\eta}$$

Then:

$$\frac{\partial P}{\partial t} = \int \frac{d^2\eta}{\pi^2} e^{\alpha\eta^*} e^{-\alpha^*\eta} \frac{\partial}{\partial t}\chi_N(\eta,\eta^*;t)$$

$$= \left[\left(\frac{\gamma}{2}+i\omega_0\right)\frac{\partial}{\partial\alpha}\alpha + \left(\frac{\gamma}{2}-i\omega_0\right)\frac{\partial}{\partial\alpha^*}\alpha^* + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]\left[\int \frac{d^2\eta}{\pi^2}\chi_N(\eta,\eta^*,t) e^{\alpha\eta^*} e^{-\alpha^*\eta}\right]$$

$\Rightarrow$ we employ $\frac{\partial}{\partial\alpha}\alpha\, e^{\alpha\eta^*} = e^{\alpha\eta^*} + \alpha\eta^* e^{\alpha\eta^*} = \frac{\partial}{\partial\eta^*}\left[\eta^* e^{\alpha\eta^*}\right]$

Hence

$$\int \frac{d^2\eta}{\pi^2} e^{\alpha\eta^*} e^{-\alpha^*\eta} \frac{\partial}{\partial t}\chi_N(\eta,\eta^*,t) = \int \frac{d^2\eta}{\pi^2}\chi_N(\eta,\eta^*,t)\begin{bmatrix}\left(\frac{\gamma}{2}+i\omega_0\right)\frac{\partial}{\partial\eta^*}\eta^* + \left(\frac{\gamma}{2}-i\omega_0\right)\frac{\partial}{\partial\eta}\eta \\ -\gamma\bar{n}\eta\eta^*\end{bmatrix}\left(e^{\alpha\eta^*} e^{-\alpha^*\eta}\right)$$

By parts

$$= \int \frac{d^2\eta}{\pi^2}\left[-\left(\frac{\gamma}{2}+i\omega_0\right)\eta^*\frac{\partial}{\partial\eta^*} - \left(\frac{\gamma}{2}-i\omega_0\right)\eta\frac{\partial}{\partial\eta} - \gamma\bar{n}\eta\eta^*\right]\chi_N(\eta,\eta^*,t)$$

Hence:

$$\boxed{\frac{\partial\chi_N}{\partial t} = \left[-\left(\frac{\gamma}{2}+i\omega_0\right)\eta^*\frac{\partial}{\partial\eta^*} - \left(\frac{\gamma}{2}-i\omega_0\right)\eta\frac{\partial}{\partial\eta} - \gamma\bar{n}\eta\eta^*\right]\chi_N}$$

This is the FPE for χ_N.

* Remember the relation between χ_N and χ (symmetrically-ordered) and χ_A (antinormally-ordered):

$$\chi(\eta) = e^{-|\eta|^2/2}\chi_N(\eta)$$

$$\chi_A(\eta) = e^{-|\eta|^2}\chi_N(\eta)$$

It's hence trivial to find the equation FPE for χ_A:

$$\boxed{\frac{\partial \chi_A}{\partial t} = \left[-\left(\frac{\gamma}{2}+i\omega_0\right)\eta^*\frac{\partial}{\partial \eta^*} - \left(\frac{\gamma}{2}-i\omega_0\right)\eta\frac{\partial}{\partial \eta} - \gamma(\bar{n}+1)\eta\eta^*\right]\chi_A}$$

so basically the same as for χ_N but changing $\bar{n} \longrightarrow \bar{n}+1$

(remember the discussion in pages 66 and 67)

Since $Q(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\eta\ \chi_A(\eta)\, e^{\alpha\eta^*}e^{-\alpha^*\eta}$

we can easily find that (following similar procedures as before)

$$\boxed{\frac{\partial Q}{\partial t} = \left[\left(\frac{\gamma}{2}+i\omega_0\right)\frac{\partial}{\partial \alpha}\alpha + \left(\frac{\gamma}{2}-i\omega_0\right)\frac{\partial}{\partial \alpha^*}\alpha^* + \gamma(\bar{n}+1)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]Q}$$

* similarly we can find $\chi(\eta,\eta^*)$ and using

$$W(\alpha,\alpha^*) = \int\frac{d^2\eta}{\pi^2}\ \chi(\eta,\eta^*)\, e^{\eta^*\alpha}e^{-\eta\alpha^*} \quad (\text{p. } 62)$$

we arrive to:

$$\boxed{\frac{\partial W}{\partial t} = \left[\left(\frac{\gamma}{2}+i\omega_0\right)\frac{\partial}{\partial \alpha}\alpha + \left(\frac{\gamma}{2}-i\omega_0\right)\frac{\partial}{\partial \alpha^*}\alpha^* + \gamma\left(\bar{n}+\frac{1}{2}\right)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]W}$$

so the same as for ρ but now with $\bar{n}+1/2$.

* As mentioned above, the Q and W representation are especially useful for those situations in which the P representation does not exist. Let's see an example of that. We are going to analyze the evolution of an initially squeezed state $|\alpha_0, r\rangle$ of the damped harmonic oscillator. We will use the Q-representation.

* But before doing that let's calculate the green function for the Fokker-Planck equation in the Q-representation:

$Q(\alpha,\alpha^*,t; \alpha_0,\alpha_0^*,0)$

To calculate this Green function we impose the initial condition

$$Q(\alpha,\alpha^*,0;\alpha_0,\alpha_0^*,0) = \delta^{(2)}(\alpha-\alpha_0)$$

* It's important to realize that while the green function in the P representation describes an oscillator that is initially in a coherent state, the green function in the Q-representation does not.

* The calculation of $Q(\alpha,\alpha^*,t;\alpha_0,\alpha_0^*,0)$ is straightforward but because it's the same as $P(\alpha,\alpha^*,t;\alpha_0,\alpha_0^*,0)$ but substituting $\bar{n}$ by $\bar{n}+1$:

$$Q(\alpha,\alpha^*,t;\alpha_0,\alpha_0^*,0) = \frac{1}{\pi(\bar{n}+1)\left[1-e^{-\gamma t}\right]} e^{-\left[\frac{|\alpha-\alpha_0 e^{-\gamma/2 t} e^{-i\omega_0 t}|^2}{(\bar{n}+1)(1-e^{-\gamma t})}\right]}$$

Remember that the Q representation for a squeezed state $|\alpha_0, r\rangle$ is (p. 66)

$$Q(\alpha,\alpha_0^*) = \frac{1}{4\pi^2 \operatorname{ch} r} e^{-\frac{1}{2}\left[\frac{(x_1-X_1)^2}{(1+e^{-2r})} + \frac{(x_2-X_2)^2}{(1+e^{2r})}\right]}$$

where $\alpha = (x_1+ix_2)/2$

$\alpha_0^* = (X_1+iX_2)/2$

* Therefore the solution $Q(\alpha,\alpha^*,t)$ at a time $t>0$ is given by:

$$Q(\alpha,\alpha^*,t) = \int d^2\lambda \; Q(\alpha,\alpha^*,t;\lambda,\lambda^*,0)\, Q(\lambda,\lambda^*,0)$$

This is again a gaussian integral (actually the product of 2 gaussian integrals) and the result is:

$$Q(\tilde{\alpha},\tilde{\alpha}^*,t) = \frac{1}{2\pi\sqrt{|\hat{M}(t)|}}\, e^{-\frac{1}{2}\vec{u}(t)^T \hat{M}(t)\vec{u}(t)}$$

(remember that we take $\alpha = e^{-i\omega_0 t}\tilde{\alpha}$, i.e. we are in the rotating frame with the freq. of the harmonic oscillator)

where $\vec{u}(t) = \begin{pmatrix} \tilde{\alpha} - \tilde{\alpha}_0 e^{-\gamma t/2} \\ \tilde{\alpha}^* - \tilde{\alpha}_0^* e^{-\gamma t/2} \end{pmatrix}$

and $\hat{M}(t) = \begin{pmatrix} -\mathrm{sh}\, 2r & \mathrm{ch}\, 2r + 1 \\ \mathrm{ch}\, 2r + 1 & -\mathrm{sh}\, 2r \end{pmatrix} e^{-\frac{\gamma t}{2}} + \begin{pmatrix} 0 & \bar{n}+1 \\ \bar{n}+1 & 0 \end{pmatrix}(1-e^{-\gamma t})$

* Once we know the Q-representation at any time, we can easily calculate averages. Remember (p. 66) that the Q-representation provides an easy way to evaluate averages of anti-normally ordered products of operators

$$\langle \hat{a}^n \hat{a}^{\dagger m} \rangle = \int d^2\alpha \; \alpha^n (\alpha^*)^m Q(\alpha,\alpha^*)$$

We can then easily calculate the variances for the quadratures.
Remember that (p. 10) the quadratures are defined as

$$X_1 = \hat{a}^+ + \hat{a}$$

$$X_2 = i[\hat{a}^+ - \hat{a}]$$

- Hence

$$(\Delta X_1)^2 = \langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle^2$$

$$(\Delta X_2)^2 = \langle \hat{X}_2^2 \rangle - \langle \hat{X}_2 \rangle^2$$

* Using the obtained expression for $Q(\alpha, \alpha^*, t)$ we obtain (again we have to solve some simple Gaussian integrals)

$$(\Delta X_1)^2 = [(e^{-2r}-1)e^{-\gamma t} + 2\bar{n}(1-e^{-\gamma t}) + 1]$$

$$(\Delta X_2)^2 = [(e^{2r}-1)e^{-\gamma t} + 2\bar{n}(1-e^{-\gamma t}) + 1]$$

Let's try to understand this evolution in some detail.

- At $t=0 \longrightarrow \Delta X_1 = e^{-r}$, $\Delta X_2 = e^{r}$ (remember the discussion in p. 15)

* Let's consider the case of zero temperature, i.e. $\bar{n}=0$.

We see then that for $t \gg 1/\gamma$

$$\left.\begin{aligned}(\Delta X_1)^2 &= 1\\ (\Delta X_2)^2 &= 1\end{aligned}\right\}$$ the squeezing gets lost

* One can easily see that $Q(\alpha, \alpha^*, t \gg \frac{1}{\gamma})$ becomes centered at $\alpha=0$ (have a look in the vector $\vec{u}$ in p. 129). Hence the amplitude of the squeezed state damps to zero and the variances in X_1 and X_2 become equal to 1, the value corresponding to the vacuum