Exercise 1 (2 Points)

Consider a dilute gas in a 3-dimensional box (with an extension in z-direction from 0 to a large height $L \gg 1$) at temperature T under the influence of the gravitational field. Use the theory of the canonical ensemble to obtain the probability $P(z)\, dz$ to find a particle in the layer $[z, z + dz]$, for $z > 0$.

Exercise 2 (3 Points)

Consider a non-interacting gas of N ultra-relativistic particles in a 3-dimensional volume V at temperature T. The Hamiltonian is given by $H = \sum_{i=1}^{N} c|\vec{p}_i|$ in this case, where c is the speed of light.

1. Calculate the partition function (Zustandssumme) Z and its logarithm. Note that the calculation is simplified by the fact that the particles do not interact.
2. Find the expectation value of the energy and compare it to that of the non-relativistic ideal gas.
3. Calculate the pressure of the gas.

Exercise 3 (5 Points)

Consider a substance with N spin-J particles at a temperature T placed in a time-independent magnetic field $\vec{B}$ which is oriented along the z-direction, $\vec{B} = B\vec{z}$. As in the other exercises, you may assume that the particles are non-interacting. Let us further assume that all the spins are in eigenstates of the z-component of the spin operator. Then the Hamiltonian of a single spin-J particle in the given magnetic field is

$$H = -\mu_z B, \quad \mu_z = g\mu_0 m_J,$$

where g is the Landé-factor of the particles, μ_0 is the Bohr-magneton, and m_J can take the values $-J, -J+1, \ldots, J-1, J$.

1. Calculate by means of the theory of the canonical ensemble the magnetization $M = N\langle\mu_z\rangle$ as a function of $\eta \equiv g\mu_0 B/(k_B T)$. The expression $\sum_{k=0}^{n} x^k = (1 - x^{n+1})/(1 - x)$ for $x \neq 1$ may help you to simplify the result.
2. Consider the small-T and large-T limits:
   - How does the magnetization behave for $T \to 0$?
   - The magnetic susceptibility of a substance χ is defined as $M = \chi|\vec{B}|$. How does χ depend on T for $T \gg 1$?

   Hint: Consider the limiting behaviour of M as a function of η. For $T \gg 1$ expand M up to terms of first order in η.