Exercise 1 (1 Point)

Consider N particles, and g boxes. How many different distributions can you find if the particles are (i) indistinguishable and fermions, or (ii) indistinguishable and bosons?

Exercise 2 (3 Points)

Consider a three-dimensional harmonic oscillator of frequency ω for a single particle of mass m.

- Consider the harmonic oscillator as classical. Calculate the canonical partition function, and from it the Helmholtz free energy, the internal energy and the entropy.
- Consider the harmonic oscillator as quantum. Calculate again the canonical partition function, and from it the Helmholtz free energy, the internal energy and the entropy.
- Compare the quantum and classical results for the free energy when $\hbar\omega/k_BT$ is very small. How do you intepret your findings?

Exercise 3 (6 Points)

Consider two particles of spin 1/2 in a magnetic field. Each one of the particles can then be in a state $|\uparrow\rangle$ or a state $|\downarrow\rangle$ (or a linear superposition of both). The Hamiltonian describing the physics of this system is provided by the expression:

$$\hat{H} = J\vec{s}_1 \cdot \vec{s}_2 + h(\hat{s}_1^z + \hat{s}_2^z) = J \sum_{i=x,y,z} s_1^i \otimes s_2^i + h(\hat{s}_1^z \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \hat{s}_2^z),$$

where $\mathbb{1}$ is the identity matrix of a single system and $\vec{s} = (\hat{s}^x, \hat{s}^y, \hat{s}^z) = \frac{1}{2}\vec{\sigma} = \frac{1}{2}(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. The entries of the last vector are the familiar Pauli matrices $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Assume $J > 0$ and $h \geq 0$.

Part A (3 of 6 Points)

- Write down the Hamiltonian in the two-particle basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, and determine its eigenvalues and eigenvectors. What is the ground state (the state of lowest energy)?
- Determine the density matrix of the possible ground states, and calculate its time evolution. Evaluate the time-average $\bar{\rho} = \frac{1}{T}\int_0^T \rho(t)$ (for $T \to \infty$).
- Consider the density matrix associated with the pure state $|\uparrow\downarrow\rangle$. Calculate its time evolution and obtain the time average.
- Consider a mixed state (remember the difference between mixed and pure state) formed by 50% of state $|\uparrow\downarrow\rangle$ and 50% of state $|\downarrow\uparrow\rangle$. Write down the corresponding density matrix, and obtain its time average.

Part B (3 of 6 Points)

The entropy of a density matrix ρ is given by $S = -k_B \cdot \text{Trace}[\rho \ln \rho]$.

- Use the fact that the trace of a matrix is independent of the representation of the matrix to show that $S = -k_B \sum_i \lambda_i \ln \lambda_i$, where λ_i are the eigenvalues of ρ.
- Calculate the entropy of both the initial state and the time averaged state for the cases (i) the initial state is the ground state, and (ii) the initial state is $|\uparrow\downarrow\rangle$, with the help of the results of part A.