Exercise 1 (4 Points)

Consider an ideal gas of H_2 molecules in the vibrational and electronic ground-state (hence the total electronic spin is zero). Since the protons have spin 1/2, the total nuclear spin can be $S = 1$ (ortho-H_2, with a symmetric wavefunction for the proton spins) and $S = 0$ (para-H_2 with an anti-symmetric wavefunction for the proton spins). Ortho- and para-H_2 differ in their rotational energies due to symmetry. The rigid-rotor Hamiltonian is provided by $\hat{H}_{rot} = \hat{L}^2/2I$, where I is the momentum of inertia of the molecule, and $\hat{L}$ is the angular momentum operator. Hence the eigenstates of $\hat{H}_{rot}$ are $E_l = \frac{\hbar^2}{2I} l(l+1)$, independently of the quantum number m (which characterizes the eigenstates of $\hat{L}_z$). Since the overall (spatial + spin) wavefunction must be antisymmetric with respect to proton interchange, only odd l is possible in ortho-H_2, and even l in para-H_2.

Consider H_2 molecules at a given temperature T using the Boltzmann statistics.

- Calculate the canonical partition function of the system.
- Calculate N_{ortho}/N_{para}, i.e. the relation between the total populations of the ortho- and para-H_2.
- What happens at large T?
- What happens at low T?

(Hint: It is crucial to calculate the correct degeneracies (including the proton spin and the orbital parts) of the states which give the same energy.)

Exercise 2 (3 Points)

- Show that the fluctuations $(\Delta n_i)^2 \equiv \langle n_i^2 \rangle - \langle n_i \rangle^2$ of the population of a given level i of energy ϵ_i in an ideal gas (Fermion, Boson or Boltzmann) are given by

$$(\Delta n_i)^2 = \frac{-1}{\beta} \frac{\partial \langle n_i \rangle}{\partial \epsilon_i},$$

- Calculate $(\Delta n_i)^2$ for fermions as function of $\{\beta, \mu, \epsilon_i\}$. Re-write the expression as an explicit function of $\langle n_i \rangle$.
- Calculate the same for a Boltzmann gas, and compare it with the result for fermions.
- Sketch $(\Delta n_i)^2$ as a function of ϵ_i. How do you interpret the results? How do the fluctuations for fermions behave when the temperature tends to zero?

Exercise 3 (3 Points)

Consider a quantum system with single-particle levels with energies ϵ_α, where α is a set of characteristic quantum numbers. Consider a fermionic gas with Spin-1/2 particles in this system, and assume that each one of the levels can be either unoccupied, or occupied by a particle with $m = 1/2$ or a particle with $m = -1/2$, but not by both $m = 1/2$ and $m = -1/2$ at the same time. As above, m is the quantum number characterizing the eigenstates of L_z.

- Calculate the grand-canonical partition function for this system.
- Calculate the average occupation $\langle n_\alpha \rangle$ of a level with energy ϵ_α.
- Sketch $\langle n_\alpha \rangle$ as a function of ϵ_α for low temperature and for large one. Compare the results with those of the standard Fermi distribution.