Exercise 1 (3 Points)

We know that for a Fermi gas $\lambda_{dB}^3/v = f_{3/2}(z)$, where

$$f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx \frac{x^2}{z^{-1}e^{x^2}+1}.$$

We are in this exercise interested in the behavior of $f_{3/2}(z)$ for large $z \gg 1$. Let $z = e^\nu$, i.e. $\nu = \beta\mu$.

- Show by using partial integration that $f_{3/2}(z)$ can be written in the form:

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} \int_0^\infty dy \frac{y^{3/2}e^{y-\nu}}{(e^{y-\nu}+1)^2}.$$

- By expanding $y^{3/2}$ around ν, and using the fact that $\nu \gg 1$, show that

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}}(I_0\nu^{3/2} + \frac{3}{2}I_1\nu^{1/2} + \frac{3}{8}I_2\nu^{-1/2} + \cdots),$$

  where $I_n \equiv \int_{-\infty}^\infty \frac{t^n e^t}{(e^t+1)^2}$.

- Show that $I_n = 0$ for odd values of n.
- Using $I_0 = 1$, $I_2 = \pi^2/3$ retrieve the expression we wrote in the lecture.

Exercise 2 (3 Points)

- Calculate the entropy of an ideal spinless Fermi gas in free space as a function of β, z and the level energies $\epsilon_{\vec{p}}$. (You may leave the sums without solving them).
- Express the entropy as a function of the mean occupations $\langle n_{\vec{p}}\rangle$. Write it in the form $S = \sum_{\vec{p}} S_{\vec{p}}$, where $S_{\vec{p}}$ is the contribution to the entropy coming from the $\vec{p}$ level.
- Draw a sketch of how $S_{\vec{p}}$ depends on $|\vec{p}|$. Which consequences do you extract from the sketch?
- Draw a plot of the total entropy S for different values of T/T_F for $T/T_F \ll 1$ (with Maple, Mathematica, ...). You should see that S goes linear with T/T_F. Why do you think is it like that? (Hint: Remember the qualitative discussion about the dependence of C_V on T that we did during the class).

Exercise 3 (4 Points)

Consider a two-dimensional non-interacting system of free electrons.

- Calculate the Fermi energy ϵ_F as a function of the surface density $\sigma = N/S$, where S is the surface area.
- Calculate the chemical potential μ as a function of T and σ, and sketch it in units of T/T_F.
- For $T/T_F \ll 1$ obtain the first correction $f(T/T_F)$ to the chemical potential for non-zero temperature: $\mu/\epsilon_F = 1 + f(T/T_F)$.