• Bose-Einstein Condensates

* Before starting with strongly-correlated gases, let's have a look to some key ideas of weakly-interacting gases (most specifically Bose gases).

. In the following, we will consider a dilute Bose gas in which the average interparticle distance is much larger than the range of the inter-atomic interactions. We may then limit ourselves to binary interactions (3-body processes are negligible).

* The Hamiltonian describing a weakly-interacting Bose gas in an external potential $V_{ext}(\vec{r})$ is:

Bosonic field operator

$$\hat{H} = \int d^3r \left\{ \frac{\hbar^2}{2m} \vec{\nabla}\hat{\psi}^+(\vec{r}) \cdot \vec{\nabla}\hat{\psi}(\vec{r}) + V_{ext}(\vec{r})\,\hat{\psi}^+(\vec{r})\,\hat{\psi}(\vec{r}) \right\}$$

$$+ \frac{1}{2}\int d^3r \int d^3r'\, \hat{\psi}^+(\vec{r})\,\hat{\psi}^+(\vec{r}\,')\, \underset{\substack{\uparrow \\ \text{interaction potential}}}{V(\vec{r}-\vec{r}\,')}\, \hat{\psi}(\vec{r}\,')\,\hat{\psi}(\vec{r})$$

* We will come back to this Hamiltonian in a moment. But first of all let's introduce the idea of <u>long-range order.</u>

Let's define the density matrix:

$$\rho(\vec{r},\vec{r}\,') = \langle \hat{\psi}^+(\vec{r})\,\hat{\psi}(\vec{r}\,')\rangle$$

We may diagonalize this matrix to get

$$\rho(\vec{r},\vec{r}\,') = \sum_i n_i\, \varphi_i^*(\vec{r})\,\varphi_i(\vec{r}\,')$$

where n_i are the eigenvalues and $\varphi_i(\vec{r})$ the eigenfunctions (<u>natural states</u>).

* We can then re-write the field operator $\hat{\psi}(\vec{r})$ in the basis of natural states.

$$\hat{\psi}(\vec{r}) = \sum_i \varphi_i(\vec{r})\,\hat{a}_i$$

where $\{\hat{a}_i\}$ are also bosonic operators (annihilating a particle in $\varphi_i(\vec{r})$).

* The <u>Bose-Einstein condensation</u> occurs below a given $T<T_c$, when there's a macroscopically large eigenvalue $n_0 = N_0$ of the order of the total number of particles.

* The corresponding natural state $\varphi_0(\vec{r})$ plays a crucial role in the BEC theory, being the so-called condensate wavefunction.
* We may then write:

$$\hat{\psi}(\vec{r}) = \varphi_0(\vec{r})\,\hat{a}_0 + \sum_{i\neq 0} \varphi_i(\vec{r})\,\hat{a}_i$$

* Now we will perform a crucial approximation, known as Bogoliubov approximation, which consists on replacing

$$\hat{a}_0,\ \hat{a}_0^+ \xrightarrow{\ \sim\ } \sqrt{N_0} \rightarrow \text{which is not any more an operator but a c-number.}$$

In doing this we are forgetting the non-commutative character of the operators. This is because $(a_0, a_0^+) = 1$, but since $\langle a_0^+ a_0\rangle = N_0 \gg 1$, then $\hat{a}_0$ and $\hat{a}_0^+$ are of order $\sqrt{N_0} \gg 1$. We can then neglect the operator character.

Note that this approximation is based on two important assumptions: first, there is a macroscopically populated natural state (and only one); second, its population is $N_0 \gg 1$.

* We can then re-write:

$$\hat{\psi}(\vec{r}) \simeq \psi_0(\vec{r}) + \delta\hat{\psi}(\vec{r})$$

where $\psi_0(\vec{r}) = \sqrt{N_0}\,\varphi_0(\vec{r}) \longrightarrow$ condensed part (c-number)

$\delta\hat{\psi}(\vec{r}) = \sum_{i\neq 0} \varphi_i(\vec{r})\,\hat{a}_i \longrightarrow$ non-condensed part (operator) (fluctuations with zero average, $\langle\delta\hat{\psi}(\vec{r})\rangle = 0$)

At very low T we can forget the non-condensed part

$$\psi(\vec{r}) \simeq \psi_0(\vec{r})$$

This approximation is crucial, since now we will deal with c-numbers only, which is of course a much simpler task!

* The function $\psi_0(\vec{r})$ is a complex quantity

$$\psi_0(\vec{r}) = |\psi_0(\vec{r})|\,\exp[iS(\vec{r})]$$

Here $|\psi_0(\vec{r})|^2$ gives the density distribution of the BEC and $S(\vec{r})$ is the phase.

* Let us come back to our Hamiltonian of page 2. The Heisenberg equation for the field operator $\hat{\psi}(\vec{r},t)$ is:

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(\vec{r},t) = [\hat{\psi}(\vec{r},t),\hat{H}] = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r},t) + \int d^3r'\,\hat{\psi}^\dagger(\vec{r}',t)V(\vec{r}-\vec{r}')\hat{\psi}(\vec{r}',t)\right\}\hat{\psi}(\vec{r},t)$$

Introducing the Bogoliubov approximation $\hat{\psi}(\vec{r},t) \longrightarrow \psi_0(\vec{r},t)$:

$$i\hbar\frac{\partial}{\partial t}\psi_0(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r},t) + \int d^3r'\,\psi_0^*(\vec{r}',t)V(\vec{r}-\vec{r}')\psi_0(\vec{r}',t)\right]\psi_0(\vec{r},t)$$

* For short-range interatomic interactions we approximate

$$V(\vec{r}-\vec{r}') \simeq g\,\delta(\vec{r}'-\vec{r}) \quad \text{with } g = \frac{4\pi\hbar^2}{m}a \quad (a = \text{s-wave scattering length})$$

and hence:

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi_0(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r},t) + g|\psi_0(\vec{r},t)|^2\right]\psi_0(\vec{r},t)}$$

which is the Gross-Pitaevskii equation (GPE), which plays a crucial role in the study of BEC at low T.

* Before moving on, let's have a look to the time-independent GPE. For this we need to come back shortly to the definition of $\psi_0(\vec{r})$.

Recall that $\hat{\psi}(\vec{r}) \simeq \psi_0(\vec{r}) + \delta\hat{\psi}(\vec{r})$

Since $\langle\delta\hat{\psi}(\vec{r})\rangle = 0 \rightarrow \psi_0(\vec{r}) = \langle\hat{\psi}(\vec{r})\rangle$ (i.e. $\psi_0(\vec{r})$ is the mean-field)

However, since $\hat{\psi}(\vec{r}) \simeq \varphi_0(\vec{r})\hat{a}_0$ (i.e. it destroys one particle) the $\langle\hat{\psi}(\vec{r})\rangle$ mean actually $\langle N-1|\hat{\psi}|N\rangle$ where actually the states $|N-1\rangle$ and $|N\rangle$, with $N-1$ and N particles respectively, are physically equivalent up to corrections of order $1/N \ll 1$.

· Let $|N\rangle$ be the stationary state for N particles

$$|N\rangle(t) = |N\rangle(0)\exp\left[-iE(N)t/\hbar\right]$$

Then:

$$\psi_0(\vec{r},t) = e^{-i[E(N)-E(N-1)]t/\hbar}\,\psi_0(\vec{r},0)$$

Note that the <u>chemical potential</u> μ is defined as the change of the energy of the system when a particle is added

$$\mu = \frac{\partial E}{\partial N} \cong E(N) - E(N-1)$$

Hence $\psi_0(\vec{r},t) = e^{-i\mu t/\hbar}\,\psi_0(\vec{r})$

- Inserting this into the GPE we get <u>the time-independent GPE</u>:

$$\boxed{\mu\,\psi_0(\vec{r}) = \left\{\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) + g|\psi_0(\vec{r})|^2\right\}\psi_0(\vec{r})}$$

Note, finally, that the chemical potential μ is fixed by the <u>normalization condition</u>: $\int |\psi_0(\vec{r})|^2\, d^3r = N$

- We will finish ~~our~~ discussion on BEC basics with an alternative formulation to the GPE. Recall that

$$\psi_0(\vec{r},t) = \sqrt{n(\vec{r},t)}\; e^{iS(\vec{r},t)} \quad \text{with } n(\vec{r},t) = |\psi_0(\vec{r},t)|^2$$

Substituting this into the GPE we get 2 equations:

(i) $\boxed{\frac{\partial}{\partial t} n = -\vec{\nabla}\cdot\vec{J}}$ where $\vec{J} = \frac{-i\hbar}{2m}\left[\psi_0^*\vec{\nabla}\psi_0 - \psi_0\vec{\nabla}\psi_0^*\right] = \frac{\hbar}{m} n\,\vec{\nabla}S$

is the <u>current</u>.

↳ Continuity equation

(ii) $$\hbar\frac{\partial}{\partial t}S + \left[\frac{m v_s^2}{2} + V_{ext}(\vec{r}) + gn \underbrace{- \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}}_{\text{quantum pressure}}\right] = 0$$

with $\vec{v}_s = \frac{\hbar}{m}\vec{\nabla}S$ → velocity of the condensate flow

For sufficiently large interactions $gn \gg \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}$ and we can neglect the quantum pressure (Thomas-Fermi regime)

- In this limit

$$\boxed{m \frac{\partial}{\partial t} \vec{v}_s + \vec{\nabla}\left[\frac{m v_s^2}{2} + V_{ext}(\vec{r}) + gn\right] = 0}$$

This equation is well-known in hydrodynamics, being the Euler equation for a nonviscous gas with pressure $P = gn^2/2$.

- For quasi-stationary configuration $\vec{v}_s \simeq 0$, and hence

$$\vec{\nabla}\left[V_{ext}(\vec{r}) + gn\right] = 0$$

$\longrightarrow V_{ext}(\vec{r}) + gn = \text{constant} \longrightarrow$ it's actually the chemical potential μ (look the time-independent GPE (p. 5) without kinetic energy)

- Then for the stationary solution:

$$\hbar \frac{\partial}{\partial t} S \simeq -\mu \longrightarrow S = -\mu/\hbar \longrightarrow$$ as we knew from p. 5.

- We will employ this so-called hydrodynamic formulation later.

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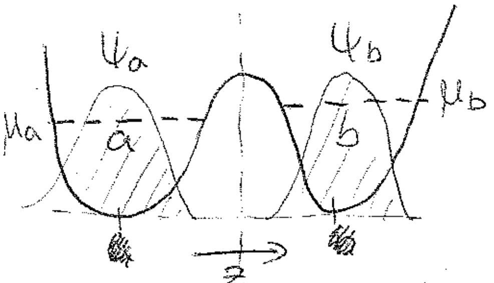
* We have now seen the basic equations (GPE or alternatively the hydrodynamic eqs.) governing the physics of BECs at very low T. It's important to understand when this formalism is not valid. A good example to analyze the limitations of the GP formalism is provided by the physics of bosons in two-well potentials.

Josephson Effect

- The Josephson effect is an important quantum phenomenon that consists of a coherent flow of particles which tunnel through a barrier in the presence of a chemical potential gradient. It has been observed in superconductors, superfluid Helium, and very recently in atomic BEC's (Oberthaler, 2005). [PRL 95, 010402(2005)]
- We will first consider the problem in a mean-field (GP-like) formalism. Let's consider an atomic Bose gas in a two-well potential.



- We assume the barrier high enough such that the overlap of the two clouds is small
- The 2 clouds have slightly different chemical potential

- For a given number of particles $N_{a,b}$ we may calculate the wavefunction $\psi_{a,b}(\vec{r}, N_{a,b}) \longrightarrow \int |\psi_{a,b}|^2 d^3r = N_{a,b} \Rightarrow N = N_a + N_b$.

The ground state is given by the symmetric superposition:

$$\psi(\vec{r},t) = \left[\psi_a(\vec{r}, N/2) + \psi_b(\vec{r}, N/2)\right] e^{i\mu t/\hbar} \quad \left(\begin{array}{c}\text{we assume}\\ \text{here } \mu_{a,b} = \mu\end{array}\right)$$

- Let's consider now non-stationary solutions describing the exchange of particles between the wells:

$$\psi(\vec{r},t) = \psi_a(\vec{r}, N_a(t))\, e^{iS_a(t)} + \psi_b(\vec{r}, N_b(t))\, e^{iS_b(t)}$$

The current density is:

$$\vec{J} = \frac{-i\hbar}{2m}\left(\psi^*(\partial_z\psi) - (\partial_z\psi^*)\psi\right) = -\frac{\hbar}{m}\left[\psi_a\partial_z\psi_b - \psi_b\partial_z\psi_a\right]\sin\Phi$$

where $\Phi = S_a - S_b$ is the <u>relative phase</u>.

- From the continuity equation (p. 5): $\partial n/\partial t = -\vec{\nabla}\cdot\vec{J}$ one may obtain:

$$\boxed{\frac{\partial N_a}{\partial t} = I_J \sin\Phi = -\frac{\partial N_b}{\partial t}} \quad \text{with} \quad \boxed{I_J = \frac{\hbar}{m}\iint dx\,dy\left[\psi_a\partial_z\psi_b - \psi_b\partial_z\psi_a\right]_{z=0}}$$

$\longrightarrow$ <u>Josephson amplitude</u> ($I_J > 0$)

* From p. 6 we know that for a quasi-stationary flow:

$$\hbar \frac{\partial S}{\partial t} \simeq -\mu \longrightarrow \boxed{\frac{\partial \phi}{\partial t} = -\frac{1}{\hbar}(\mu_a - \mu_b)}$$

* Let's introduce $K = \frac{N_a - N_b}{2} \longrightarrow \left.\begin{array}{l} N_a = N/2 + K \\ N_b = N/2 - K \end{array}\right\}$ Then K is the deviation from equilibrium

We assume only slight imbalances and hence $K << N/2$

Then $$\mu_a = \mu(N_a) \simeq \mu(N/2) + \left(\frac{\partial \mu}{\partial N_a}\right)_{N/2} K$$

$$\mu_b \simeq \mu(N/2) - \left(\frac{\partial \mu}{\partial N}\right)_{N/2} K$$

Hence $$\frac{\partial \Phi}{\partial t} \cong -\frac{1}{\hbar}\left(\frac{\partial \mu}{\partial N}\right)_{N/2} 2K = -\frac{E_c}{\hbar} K \longrightarrow E_c = 2\left(\frac{\partial \mu}{\partial N}\right)_{N/2}$$

* Moreover, since $\frac{\partial N_a}{\partial t} = -\frac{\partial N_b}{\partial t} = I_J \sin\Phi \rightarrow \frac{\partial K}{\partial t} = I_J \sin\Phi$
* We have hence two equations describing the population imbalance and the relative phase, which are coupled with each other:

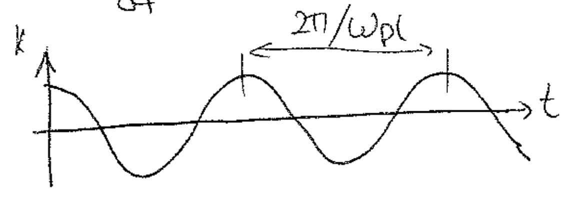
$$\boxed{\begin{array}{l} \frac{\partial \Phi}{\partial t} = -\frac{E_c}{\hbar} K \\ \frac{\partial K}{\partial t} = \frac{E_J}{\hbar} \sin\phi \end{array}}$$

where $E_J = \hbar I_J \equiv$ Josephson energy.

$\longrightarrow$ These are exactly the equations of a pendulum, and the physics is hence quite easy to understand.

* If K and Φ are small $\rightarrow \sin\phi \simeq \phi \rightarrow \frac{\partial K}{\partial t} = \frac{E_J}{\hbar}\phi$

$$\rightarrow \frac{\partial^2 K}{\partial t^2} = -\omega_{pl}^2 K \quad \text{with} \quad \omega_{pl} = \frac{1}{\hbar}\sqrt{E_c E_J} \equiv \text{plasma freq.}$$



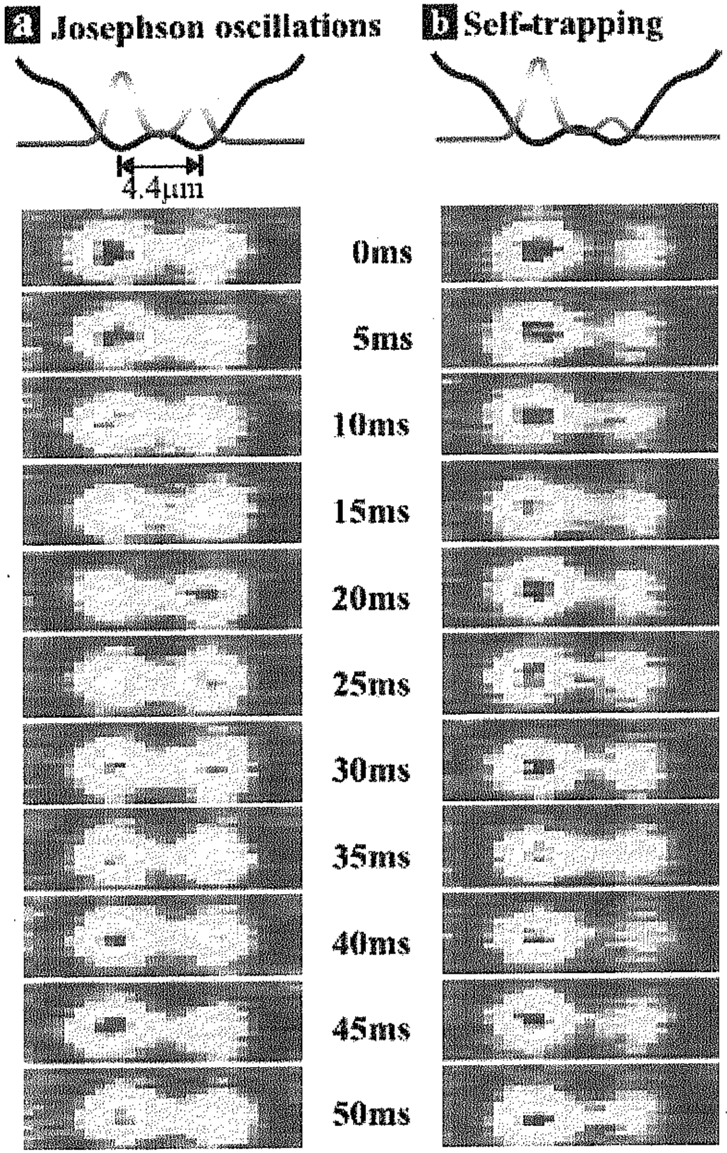
Josephson oscillations

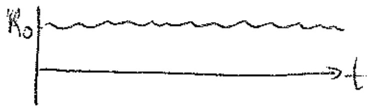
The population oscillates between both wells.

* On the contrary if $K(t=0) >> E_J/E_c$, $K \simeq K(0)$ for all times $\quad \tau = E_J t/\hbar$

since $$K(\tau) = K(0) + \int_0^\tau d\tau' \underbrace{\sin\left[\phi(0) - \frac{E_c}{E_J}\int_0^{\tau'} K(\tau'')d\tau''\right]}_{\text{oscillates very fast and averages to zero}}$$

Josephson experiment of M. Oberthaler [Albiez et al., PRL **95**, 010402 (2005)]



* This is the so-called <u>self-trapping</u> 

* We may re-write the pendulum eqs. in a convenient form useful for the discussion below

$$\left.\begin{aligned}\frac{\partial \Phi}{\partial t} &= \frac{\partial H_J}{\partial(\hbar k)} \\ \frac{\partial(\hbar k)}{\partial t} &= -\frac{\partial H_J}{\partial \Phi}\end{aligned}\right\} \text{ with } \boxed{H_J = \frac{E_c}{2}k^2 - E_J\cos\phi} \rightarrow \text{Josephson Hamiltonian.}$$

These are Hamilton equations where Φ and $\hbar k$ are canonically conjugated variables.

* We may now introduce canonical quantization:

$$\left.\begin{aligned}\Phi &\longrightarrow \hat{\Phi} \\ \hbar k &\longrightarrow \hbar\hat{k}\end{aligned}\right\} [\hat{\Phi}, \hbar\hat{k}] = i\hbar \rightarrow [\hat{\phi}, \hat{k}] = i \rightarrow \hat{k} = -i\frac{\partial}{\partial\Phi}$$

(in Φ-representation)

Then the <u>Josephson Hamiltonian</u> becomes:

$$\boxed{\hat{H}_J = -\frac{E_c}{2}\frac{\partial^2}{\partial\Phi^2} - E_J\cos\Phi}$$

* Due to the periodicity constraint, the uncertainty relation obeyed by the fluctuations of $\hat{\phi}$ and $\hat{k}$ takes the form:

$$(\Delta K)^2\left[\Delta(\sin(\phi-\phi_0))\right]^2 \geq \left|\frac{i}{2}\langle[K, \sin(\phi-\phi_0)]\rangle\right|^2 = \frac{1}{4}\langle\cos(\phi-\phi_0)\rangle^2$$

(where $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$). This holds for arbitrary ϕ_0.

* Only if $\phi_0 = 0$ we may then approximate $(\Delta K)^2(\Delta\phi)^2 \geq 1/4$, the usual form we know.

* The <u>coherence factor</u> $\alpha = \langle\cos(\phi-\phi_0)\rangle$ characterizes the degree of phase coherence of the system:
  * $\alpha = 1 \rightarrow$ full coherence $\rightarrow \phi$ is localized around ϕ_0
  * $\alpha = 0 \rightarrow$ absence of coherence $\rightarrow \phi$ is arbitrary

* In the <u>strong-tunneling limit</u> $E_C/E_J \ll 1$: $\phi \simeq 0$ and

$$\hat{H}_J \simeq -\frac{E_C}{2}\frac{\partial^2}{\partial\phi^2} + E_J\phi^2/2 \quad \text{(apart from an unimportant constant)}$$

→ as we already mentioned, this is an harmonic oscillator with frequency $\omega_{pl} = \sqrt{E_C E_J}/\hbar$. The ground state is hence

$$\psi(\phi) = \frac{1}{\sqrt{\pi}\, l_{HO}} e^{-\phi^2/2l_{HO}^2} \quad \text{with } l_{HO} = \left(E_C/E_J\right)^{1/4}$$

Hence → $(\Delta\phi)^2 = \langle\phi^2\rangle = \int d\phi\, \phi^2 |\psi(\phi)|^2 = \frac{1}{2}\sqrt{\frac{E_C}{E_J}} \ll 1$ } → phase fluctuations are, as expected, small

Since for $\phi \simeq 0 \to (\Delta K)^2 = \frac{1}{4(\Delta\phi)^2}$ in the ground state, then $(\Delta K)^2 \gg 1$

→ <u>large relative number fluctuations</u>.

- On the contrary, for <u>the weak-tunneling limit</u> $E_C/E_J \gg 1$, we may approximate:

$$\hat{H}_J \simeq -\frac{E_C}{2}\frac{\partial^2}{\partial\phi^2} \longrightarrow \left.\begin{matrix}\text{Eigenstates}\\ \text{are plane waves}\end{matrix}\right\} e^{in\phi} \quad \begin{matrix}(n \in \mathbb{Z})\\ n \neq 0\end{matrix} \longrightarrow |\psi(\phi)| \text{ is constant}$$

→ <u>the relative phase is arbitrary!</u> → the relative number is fixed ($\Delta K \to 0$)

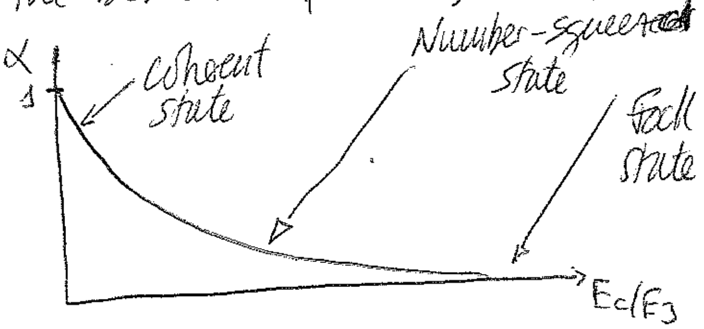
→ <u>the ground state is the Fock state</u> $|K=0\rangle$

Introducing the first-order corrections due to the $-E_J\cos\phi$ term one finds:

$$\alpha \simeq 2\, E_J/E_C$$

$$(\Delta K)^2 \simeq 2\left(E_J/E_C\right)^2$$

- The behavior of the system as a function of E_C/E_J is summarized as follows:



* the GP formalism is only OK when the system is phase coherent.
* Hence, when tunneling decreases the GP-formalism becomes inadequate!!

[• <u>Note</u>: For a fixed E_C/E_J coherence may be also lost due to thermal phase fluctuations. This may be employed for thermometry as shown in the group of M. Oberthaler.]
↳ [PRL 96, 130404 (2006)]

* let's see now this problem from an alternative perspective, which will be later quite useful for our discussion of lattice gases.

We come back to the Hamiltonian of page 2:

$$\hat{H} = \int d^3r\, \hat{\psi}^+(\vec{r}) \left[\frac{-\hbar^2\nabla^2}{2m} + \underbrace{V_{ext}(\vec{r})}_{\text{double-well potential}} \right] \hat{\psi}(\vec{r}) + \frac{g}{2} \int d^3r\, \hat{\psi}^+(\vec{r})\hat{\psi}^+(\vec{r})\hat{\psi}(\vec{r})\hat{\psi}(\vec{r})$$

* We employ the ansatz: $\hat{\psi}(\vec{r}) = \phi_a(\vec{r})\, \underbrace{\hat{a}}_{\text{left well}} + \phi_b(\vec{r})\, \hat{b}$ ⟶ right well

$\phi_{a,b}(\vec{r}) \equiv$ ground-state single-particle wavefunctions for the 2 separate wells.

* In the quasi-ideal regime, due to hopping, the symmetric state $\phi_0 = \frac{(\phi_a + \phi_b)}{\sqrt{2}}$ is the ground state, whereas the antisymmetric one $\phi_1 = (\phi_a - \phi_b)/\sqrt{2}$ is a excited state.
* Inserting the ansatz in the Hamiltonian (assuming a negligible overlapping between the wells) we get the tight-binding Hamiltonian:

$$\boxed{\hat{H}_{BH} \cong \frac{E_C}{4}(a^+a^+aa + b^+b^+bb) - \frac{\delta_J}{2}(a^+b + b^+a)}$$

⟶ BOSE-HUBBARD HAMILTONIAN (we'll find it again in our discussion of opt. lattices)

with $E_C = 2g\int |\phi_a|^4 d^3r$

$$\delta_J = -2\int d^3r\, \phi_a^*(\vec{r}) \left(\frac{-\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r}) \right) \phi_b(\vec{r})$$

* let $\hat{a}_0 = (\hat{a} + \hat{b})/\sqrt{2}$ and $\hat{a}_1 = (\hat{a} - \hat{b})/\sqrt{2}$
* As in p. 3 we employ Bogoliubov approximation: $\hat{a}_0, \hat{a}_0^+ \simeq \sqrt{N}$ assuming BEC in ϕ_0. We re-write the $\hat{H}_{BH}$ retaining up to quadratic terms in $\hat{a}_1, \hat{a}_1^+$:

$$\hat{H}_{BH} \simeq \delta_J \hat{a}_1^+ a_1 + \frac{E_C}{8} N (\hat{a}_1^+\hat{a}_1^+ + \hat{a}_1\hat{a}_1 + 2\hat{a}_1^+\hat{a}_1)$$

We employ the Bogoliubov transformation

$$\left.\begin{aligned} \hat{a}_1 &= u\hat{\alpha} + v\hat{\alpha}^+ \\ \hat{a}_1^+ &= v\hat{\alpha} + u\hat{\alpha}^+ \end{aligned}\right\} \quad \text{with } u,v = \pm\sqrt{\frac{\delta_J + NE_C/4}{2\epsilon_J} \pm 1/2}$$

$$\epsilon_J = \sqrt{\delta_J(\delta_J + NE_C/2)}$$

Hence (up to a constant):

$$\hat{H} = \epsilon_J\, \hat{\alpha}^+\hat{\alpha}$$

* The key point here is the so-called quantum depletion of the condensate:

$$\delta N_0 = \vartheta^2 = \frac{\delta_J + N E_c/4}{\epsilon_J} - 1/2 \qquad \left(\begin{array}{l} \text{since } N_0 + N_1 = N, \text{ then } \delta N_0 \\ \text{is the population in } \phi_1 \end{array} \right)$$

The Bogoliubov approximation (and hence the GP analysis) is just valid if $\delta N_0 \ll N \longrightarrow E_c/E_J \ll 1 \rightarrow$ i.e. the coherent regime of p. 10.

* When $E_c/E_J \gtrsim 1 \longrightarrow$ fragmentation occurs
  ↳ both $\phi_{0,1}$ are macroscopically populated

$\Longrightarrow$ We can't use the GP-formalism any more.

* In summary, the GP formalism is just valid if the condensate depletion is small, which coincides with ~~the~~ phase coherent regime. The GP-formalism fails when quantum fluctuations deplete the condensate, which coincides with the the phase incoherent (number-squeezed) regime.

(• Note: thermal fluctuations may lead as well to a breakdown of the GP formalism, but here we are less concern about them)

* For the two-well case quantum fluctuations dominate when the hopping becomes small. For optical lattices (as we will see) it's the same. But this is a general observation: if quantum fluctuations dominate, then mean-field treatments à la GP-formalism are doomed to fail.