ATOMS IN OPTICAL LATTICES (III): OTHER LATTICE MODELS AND CONTROL KNOBS

- Up to now, we just saw the case of a 2-component Bose or Fermi gas, showing that it can be mapped for $U \gg t$ to a spin Hamiltonian due to Super-exchange.
- We are going to leave a bit in this lecture about other ways of mapping ultra-cold atoms into spin models, as well as about other control knobs that cold lattice gases allow.

* HARD-CORE BOSONS: MAPPING OCCUPATION INTO SPIN

- Let's consider the case of single-component bosons in an optical lattice

$$\hat{H} = -t \sum_{\langle ij \rangle} \hat{b}_i^\dagger \hat{b}_j + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1) - \mu \sum_i \hat{n}_i$$

- Let's consider the case $U \gg t$ such that double occupancy of any site is energetically very costly. In that case we may assume $(\hat{b}_i^\dagger)^2 = 0$, and hence consider that $n_i = 0, 1$ in all sites. This is the so-called hard-core regime.

- We may then map the bosonic system into a spin model (with $S = 1/2$)

$$\hat{S}_i^+ = \hat{b}$$
$$\hat{S}_i^- = \hat{b}^\dagger$$
$$\hat{S}_i^z = \frac{1}{2} - \hat{n}$$

- Note that this is nothing else than the Holstein-Primakoff transformation of p.62, assuming $S = 1/2$ and $(\hat{b}_i^\dagger)^2 = 0$
- Hence in the spin language: $|n=0\rangle \Rightarrow |\uparrow\rangle$, $|n=1\rangle \Rightarrow |\downarrow\rangle$ } We map occupation into spin

- Then:

$$\hat{H} = -t \sum_{\vec{\delta}} \sum_{\vec{\sigma}} \left(\hat{b}_{\vec{\delta}}^\dagger \hat{b}_{\vec{\delta}+\vec{\sigma}} + \hat{b}_{\vec{\delta}+\vec{\sigma}}^\dagger \hat{b}_{\vec{\delta}} \right) - \mu \sum_{\vec{\delta}} \hat{n}_{\vec{\delta}}$$

Note that the interactions disappear from the Hamiltonian since we forbid double occupancy

$$= -t \sum_{\vec{\delta}} \sum_{\vec{\sigma}} \left[\hat{S}_{\vec{\delta}}^- \hat{S}_{\vec{\delta}+\vec{\sigma}}^+ + \hat{S}_{\vec{\delta}+\vec{\sigma}}^- \hat{S}_{\vec{\delta}}^+ \right] - \mu \sum_{\vec{\delta}} \left(\frac{1}{2} - \hat{S}_{\vec{\delta}}^z \right)$$

- Eliminating constants:

$$\hat{H} = \underbrace{-2t \sum_{\vec{\delta}} \sum_{\vec{\sigma}} \left(\hat{S}_{\vec{\delta}}^x \hat{S}_{\vec{\delta}+\vec{\sigma}}^x + \hat{S}_{\vec{\delta}}^y \hat{S}_{\vec{\delta}+\vec{\sigma}}^y \right)}_{\text{The hopping term becomes a ferromagnetic XX Heisenberg Hamiltonian}} + \underbrace{\mu \sum_{\vec{\delta}} \hat{S}_{\vec{\delta}}^z}_{\text{The chemical potential term provides an effective magnetic field}}$$

$\downarrow$
$J = -2t$

* Interestingly, the sign of t (and hence the FM/AFM character of the XX Hamiltonian) may be modified by means of a <u>periodic modulation of the lattice Hamiltonian</u> [Eckardt et al., PRL 95, 260404 (2005)]. Let's have a brief look to the idea (we restrict to a 1D case).

* Let's consider a <u>periodically forced Bose-Hubbard Hamiltonian</u> described by the explicitely time-dependent form:

$$\hat{H}(t) = \hat{H}_0 + K\cos(\omega t)\sum_j j\,\hat{n}_j$$ where $\hat{H}_0$ is the Hamiltonian of page (68).

* The oscillatory term can be realized experimentally by periodically shifting the position of a mirror employed to generate the standing laser wave forming the optical lattice, and transforming to the co-moving reference frame. [*Note: If we shake the lattice as $x(t) = \Delta x \cos\omega t$, in the lattice frame we obtain an inertial force: $F = -m\ddot{x} = m\omega^2 \Delta x \cos\omega t$, and hence an oscillating on-site energy $V_j = -x_j \cdot F$ $= [-j\Delta x m\omega^2]\cos\omega t\; j = K\cos\omega t\; j$, as above.]

* Note that $\hat{H}(t) = \hat{H}(t+T)$ with $T = 2\pi/\omega$. The analysis of the problem is hence done better by means of <u>quantum Floquet theory</u>, which resembles Bloch theory of periodic potentials but now in time instead of space:

We write the solutions of the time-dependent many-body Schrödinger equation in the form:

$$|\psi_n(t)\rangle = |u_n(t)\rangle \exp\{-i\varepsilon_n t/\hbar\} \quad \text{with} \quad |u_n(t)\rangle = |u_n(t+T)\rangle$$

$\Downarrow$ <u>Floquet states</u>.

* The Floquet states are obtained by solving the eigenvalue equation:

$$[\hat{H}(t) - i\hbar\partial_t]\,|u_n(t)\rangle = \varepsilon_n |u_n(t)\rangle$$

with ε_n the <u>quasi-energies</u> defined up to an integer multiple of $\hbar\omega$ (like quasimomenta in Bloch's theory).

* If $|u_n(t)\rangle$ solves the eigenvalue equation with eigenvalues ε_n, then $|u_n(t)\rangle e^{im\omega t}$ is a T-periodic eigensolution with energy $\varepsilon_n + m\hbar\omega$ ($m = 0, \pm 1, \pm 2, \ldots$). The quasienergy spectrum has hence a <u>Brillouin-zone-like</u> structure (the width of the Brillouin zone is $\hbar\omega$).

* We introduce the Floquet basis:

$$|\{n_j\},m\rangle = |\{n_j\}\rangle \exp\left[-i\frac{K}{\hbar\omega}\sin\omega t\sum_j j n_j + im\omega t\right]$$

Fock state with n_j particles at site j

* The eigenvalue problem refers to an extended Hilbert space of T-periodic functions, in which the time is regarded as a coordinate. We hence introduce the scalar product:

$$\langle\langle\cdot|\cdot\rangle\rangle = \frac{1}{T}\int_0^T dt\,\langle\cdot|\cdot\rangle$$ (i.e. usual scalar product combined with time-averaging within one period)

To solve the eigenvalue problem we compute the matrix elements of $\hat{H}(t) - i\hbar\partial_t$ in the Floquet basis $\{|\{n_j\},m\rangle\}$ with respect to this scalar product.

- let $\hat{H}_0 = \hat{H}_{TUN} + \hat{H}_{INT}$ with $\hat{H}_{TUN} = -t\sum_{\langle ij\rangle} b_i^\dagger b_j$

Note that $-i\hbar\partial_t|\{n_j\},m\rangle = [-K\cos\omega t\sum_j n_j + im\omega]|\{n_j\},m\rangle$.

This removes the periodic part of $\hat{H}(t)$ and hence:

$$\langle\langle\{n_j'\},m'|\hat{H} - i\hbar\partial_t|\{n_j\},m\}\rangle\rangle = \langle\langle\{n_j'\},m'|\hat{H}_{TUN} + \hat{H}_{INT} + im\omega|\{n_j\},m\rangle$$

* let's have a look to the hopping part:

$$\langle\langle\{n_j'\},m'|\hat{H}_{TUN}|\{n_j\},m\rangle\rangle = \frac{1}{T}\int_0^T dt\,\langle\{n_j'\}|\hat{H}_{TUN}|\{n_j\}\rangle e^{+i\frac{K}{\hbar\omega}\sin\omega t\sum_j(n_j'-n_j)} e^{i(m-m')\omega t}$$

Since $\hat{H}_{TUN}$ just transfers one particle from a site to a neighboring one then $\sum_j (n_j'-n_j)j = \pm 1 \equiv s$.

Then:

$$\langle\langle\{n_j'\},m'|\hat{H}_{TUN}|\{n_j\},m\rangle\rangle = \frac{1}{T}\int_0^T dt\,\langle\{n_j'\}|\hat{H}_{TUN}|\{n_j\}\rangle e^{i(m-m')\omega t} e^{i\frac{Ks}{\hbar\omega}\sin\omega t}$$

$$= \langle\{n_j'\}|\hat{H}_{TUN}|\{n_j\}\rangle \underbrace{\frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{i(m-m')\theta} e^{i\frac{Ks}{\hbar\omega}\sin\theta}}_{J_{m-m'}\left[\frac{Ks}{\hbar\omega}\right] = s^{m-m'}J_{m-m'}\left(\frac{K}{\hbar\omega}\right)}$$

Bessel function $\longrightarrow$

* On the other hand

$$\langle \{n_j'\}, m' | H_{INT} + i\hbar\omega | \{n_j\}, m \rangle = \delta_{m'm} \left[\langle \{n_j'\} | \hat{H}_{INT} | \{n_j\} \rangle + m\hbar\omega \right]$$

Then:

$$\langle\langle \{n_j'\}, m' | \hat{H}(t) - i\hbar\partial_t | \{n_j\}, m \rangle\rangle = \delta_{m'm} \left[m\hbar\omega + \langle \{n_j'\} | J_0\left(\frac{K}{\hbar\omega}\right) \hat{H}_{INT} | \{n_j\} \rangle \right]$$

$$+ (1-\delta_{m'm}) S^{m'-m} J_{m'-m}\left(\frac{K}{\hbar\omega}\right) \langle \{n_j'\} | \hat{H}_{INT} | \{n_j\} \rangle$$

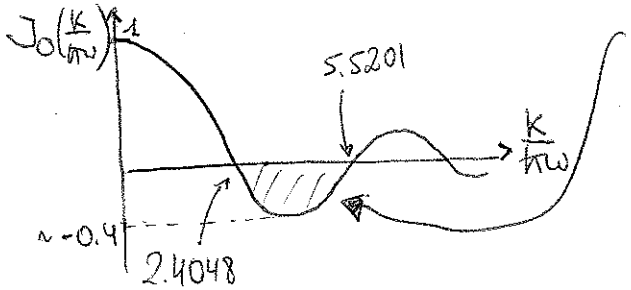
* We get hence diagonal blocks (with $m'=m$) in which we recover the original Hamiltonian but now $t \to t\, J_0(K/\hbar\omega)$. Each block is separated from a neighboring one by an energy $\hbar\omega$.

The blocks are coupled by non-diagonal terms proportional to $J_{m'-m}\left(\frac{K}{\hbar\omega}\right)$. These couplings may be neglected if $\hbar\omega$ is much larger than any other energy scale [$\hbar\omega \gg t, U$].

* If $\hbar\omega$ is so we get hence something remarkable. As mentioned above we recover the original Hamiltonian but now

$$\boxed{t \longrightarrow t\, J_0\left(\frac{K}{\hbar\omega}\right)}$$

but recall that the Bessel function J_0 behaves as:



We may hence modify the sign of the hopping constant(!!)

⇓

We may change the character of the spin model from FM to AFM!

* Hence, hard core bosons (with the "shaking" trick) allow for the simulation of ferromagnetic and antiferromagnetic planar spins.

* This trick was first experimentally realized by the group of O. Morsch/E. Arimondo in Pisa [Lignier et al, PRL 99, 220403 (2007)], and more recently in the group of K. Sengstock in Hamburg. [Struck et al., Science 333, 996 (2011)]

* POLAR LATTICE GASES

* Up to now, we have considered that the particles interact via short-range isotropic potentials (van der Waals-like). There is however a new generation of experiments in which another type of interaction, namely the dipole-dipole interaction plays a fundamental role. This is in particular the case of polar molecules (also the case of highly magnetic atoms as Chromium and Dysprosium).

* Let's discuss briefly the case of polar molecules, since they may present high electric dipole moments (much larger than atomic magnetic dipoles).

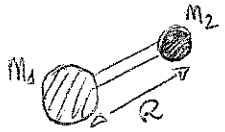
A polar molecule is maximally polar when placed into its lowest ro-vibrational state. However, although a molecule may exhibit a permanent dipole moment $\hat{d}$ in the molecular frame, it must be oriented in the laboratory frame by an external field.

This orientation may be understood from a simple rigid-rotor model of diatomic molecules:

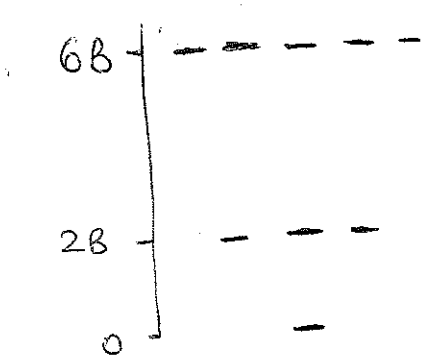
$$\hat{H}_{rot} = B\hat{J}^2$$

where $\hat{J} \rightarrow$ the molecule angular momentum operator (in $\hbar$ units)

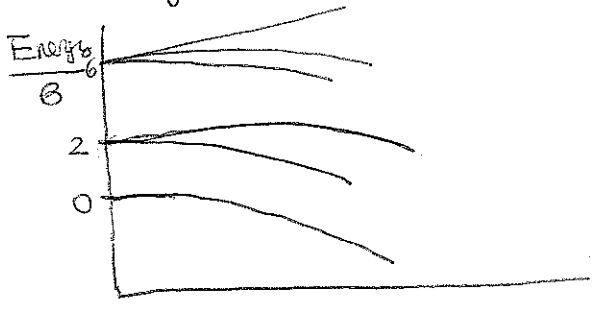
$B = \frac{\hbar^2}{2\mu R^2} \equiv$ rotational constant ($R \equiv$ equilibrium internuclear distance, $\mu \equiv$ reduced mass)



The eigenstates of $\hat{H}_{rot}$ are the rotational states $|J, m_J\rangle$ with energy $BJ(J+1)$, which are $2J+1$ times degenerate.

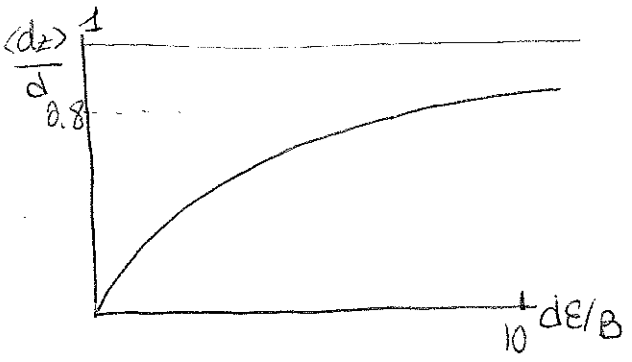


* This degeneracy is lifted in the presence of an external field $\vec{E} = \mathcal{E}\vec{e}_z$, which leads to a Stark shift $\hat{H} = \hat{H}_{rot} - d\mathcal{E}\cos\theta$, where θ is the angle between z and the internuclear axis.



* We may then obtain the average dipole moment (now in the lab frame)

$$\langle d_z \rangle = d\langle \cos\theta \rangle = -\frac{d}{dE} E_0 \quad \text{where } E_0 \equiv \text{ground-state energy}$$



* The dipole moment increases linearly for small dE/B, saturating asymptotically for $dE/B \gg 1$ toward the base value d.
* For typical values $d \sim 1D$, $B/h \sim 10$ GHz, $dE/B \sim 1$ for 10^4 V/cm which is large but experimentally accessible.

* As for this moment, polar molecules in the vibrational ground state have been produced (JILA, Heidelberg), but not yet brought to quantum degeneracy.

* As a bottom-line we should keep in mind that polar molecules allow two control "knobs":
  * The strength of the orienting field controls the dipole strength, which may range from zero to a rather large value of up to few Debyes in some molecules (1 Debye $= 3.3 \times 10^{-30}$ Cm).
  * The direction of the orienting field determines the orientation of the dipoles.

* Let's have now a look to the dipole-dipole interaction between two dipoles (oriented say along z):

$$V_{dd}(\vec{r}) = \frac{d^2}{4\pi\varepsilon_0}\left(\frac{1-3\cos^2\theta}{r^3}\right)$$

The dipole-dipole interaction presents two key features when compared with the van-der-Waals-like interaction we have assumed up to now:

* It is long-range: it depends as $\frac{1}{r^3}$ and not as $\frac{1}{r^6}$ as the usual vdW.
* It is anisotropic due to the $(1-3\cos^2\theta)$ factor

* Let's try to understand now which consequences may have the dipole-dipole interaction for the physics of lattice gases, which is our main interest here.

We consider polar bosonic particles in an optical lattice. The system is described by the Hamiltonian (recall p. 23):

$$\hat{H} = \int d^3r\, \hat{\psi}^+(\vec{r}) \left[\frac{-\hbar^2}{2m} \nabla^2 + V_{latt}(\vec{r}) + \frac{1}{2} \int d^3r'\, U(\vec{r}-\vec{r}')\, \hat{\psi}^+(\vec{r}')\hat{\psi}(\vec{r}') \right] \hat{\psi}(\vec{r})$$

(lattice)

where $U(\vec{r}) = g\,\delta(\vec{r}) + V_{dd}(\vec{r})$

[Note: The actual inter-particle interaction may be substituted by the simple pseudopotential $U(\vec{r})$, where $g = \frac{4\pi\hbar^2}{m} a$, but in principle $a = a(d)$. This dependence may manifest itself quite dramatically in the so-called shape-resonances, at which a diverges. The previous pseudopotential has been shown to be valid even away from shape resonances.]

* We may now proceed exactly as in p. 24, i.e. we expand in the basis of Wannier functions

$$\hat{\psi}(\vec{r}) = \sum_{\vec{j}} w_{\vec{j}}(\vec{r})\, \hat{a}_{\vec{j}}$$

and assume tight-binding (i.e. very poorly overlaps between wavefunctions at different sites).

* If we do so we find an effective lattice Hamiltonian which has again a Bose-Hubbard form but with inter-site interactions

$$\boxed{\hat{H} = -t \sum_{\langle j,j' \rangle} \hat{a}_j^+ \hat{a}_{j'} + \frac{U_0}{2} \sum_j \hat{n}_j(\hat{n}_j - 1) + \sum_{\delta, |\delta|>0} \frac{U_\delta}{2} \sum_j \hat{n}_j \hat{n}_{j+\delta}}$$

extended Hubbard model

where

$$t \equiv -\int d^3r\, w_j^*(\vec{r}) \left[\frac{-\hbar^2}{2m} \nabla^2 + V_{latt}(\vec{r}) \right] w_{j'}(\vec{r})$$

$$U_0 \equiv g \int d^3r\, |w_j(\vec{r})|^4 + \iint d^3r\, d^3r'\, |w_j(\vec{r})|^2 |w_j(\vec{r}')|^2 V_{dd}(\vec{r}-\vec{r}')$$

$$U_\delta \equiv \iint d^3r\, d^3r'\, |w_j(\vec{r})|^2 |w_{j+\delta}(\vec{r}')|^2 V_{dd}(\vec{r}-\vec{r}')$$

* We should compare this Hamiltonian with the Bose-Hubbard Hamiltonian of p. 24

- As in these, we have <u>on-site interactions</u>. Note however that they result from the <u>interplay between short-range and dipolar interactions</u>: $U_0 \equiv U_0^{(sc)} + U_0^{(dd)}$

- The main new ingredient is the presence of <u>intersite interactions</u>, which result exclusively from the dipole-dipole interaction.

(<u>Note</u>: A caveat here. In the derivation of the extended Hubbard model we have forgotten some terms which may become relevant for large dipoles, in particular collisionally-induced tunneling terms of the form $a_j^\dagger(\hat{n}_j+\hat{n}_{j+1})\hat{a}_{j+1}$.]

* Few further comments about <u>the "control knobs"</u> concerning the extended Bose-Hubbard model:

* Due to the anisotropy of the dipolar interaction, the <u>spatial dependence of the on-site wavefunction</u> may significantly modify $U_0^{(dd)}$. One may e.g. get $U_0 \simeq 0$ but $U_{\delta>0} \neq 0$ by playing with this (and perhaps also Feshbach resonances).

* In 2D and 1D lattices one may modify <u>the orientation of the dipoles</u> with respect to the plane (line), hence modifying the amplitude and even the sign of U_δ.

* Note that U_δ decays with δ (for zero-dimensional sites as $1/\delta^3$). This means that under proper conditions of lattice spacing and relatively weak dipole one may reduce just <u>to nearest neighbors</u> ($\delta = 1$).

(<u>Note</u>: However for large dipoles interaction to many neighbors may (and do!) play an important role.)

* Let's focus on the case with just <u>nearest-neighbor interactions</u>

$$\hat{H} = -t \sum_{\vec{j},\vec{\delta}} \left(\hat{b}^{+}_{\vec{j}} \hat{b}_{\vec{j}+\vec{\delta}} + \hat{b}^{+}_{\vec{j}+\vec{\delta}} \hat{b}_{\vec{j}} \right) + \frac{U_0}{2} \sum_{\vec{j}} \hat{n}_{\vec{j}} \left(\hat{n}_{\vec{j}} - 1 \right) + \frac{U_1}{2} \sum_{\vec{j},\vec{\delta}} \hat{n}_{\vec{j}} \hat{n}_{\vec{j}+\vec{\delta}} - \mu \sum_{\vec{j}} \hat{n}_{\vec{j}}$$

As in p. (68) we will now consider the hard-core case ($U_0 \to \infty$). We may now proceed as in p. (68) and map the hard-core case into a Spin model.

The new term becomes:

$$\frac{U_1}{2} \sum_{\vec{j},\vec{\delta}} \hat{n}_{\vec{j}} \hat{n}_{\vec{j}+\vec{\delta}} = \frac{U_1}{2} \sum_{\vec{j},\vec{\delta}} \left(\frac{1}{2} - \hat{S}^{z}_{\vec{j}} \right) \left(\frac{1}{2} - \hat{S}^{z}_{\vec{j}+\vec{\delta}} \right)$$

$$= \underbrace{\frac{U_1}{8} \sum_{\vec{j},\vec{\delta}}}_{\text{constant}} - \underbrace{\frac{U_1}{2} \sum_{\vec{j},\vec{\delta}} \left(\hat{S}^{z}_{\vec{j}} + \hat{S}^{z}_{\vec{j}+\vec{\delta}} \right)}_{\substack{\text{this regularizes} \\ \text{the chemical potential} \\ \text{(i.e. the effective magnetic field)}}} + \underbrace{\frac{U_1}{2} \sum_{\vec{j},\vec{\delta}} \hat{S}^{z}_{\vec{j}} \hat{S}^{z}_{\vec{j}+\vec{\delta}}}_{\substack{\Downarrow \\ \text{This is the truly} \\ \text{important new contribution.}}}$$

* Eliminating constants and regularizing the chemical potential we get:

$$\hat{H} = J \sum_{\vec{j},\vec{\delta}} \left(\hat{S}^{x}_{\vec{j}} \hat{S}^{x}_{\vec{j}+\vec{\delta}} + \hat{S}^{y}_{\vec{j}} \hat{S}^{y}_{\vec{j}+\vec{\delta}} \right) + \frac{U_1}{2} \sum_{\vec{j},\vec{\delta}} \hat{S}^{z}_{\vec{j}} \hat{S}^{z}_{\vec{j}+\vec{\delta}} + \mu \sum_{\vec{j}} \hat{S}^{z}_{\vec{j}}$$

$$\boxed{\hat{H} = J \sum_{\vec{j},\vec{\delta}} \left(\hat{\vec{S}}_{\vec{j}} \cdot \hat{\vec{S}}_{\vec{j}+\vec{\delta}} \right)_{\Delta} + \mu \sum_{\vec{j}} \hat{S}^{z}_{\vec{j}}} \quad \rightarrow \underline{\text{XXZ Heisenberg model}}$$

where $\left(\hat{\vec{S}}_{\vec{j}} \cdot \hat{\vec{S}}_{\vec{j}+\vec{\delta}} \right)_{\Delta} = \hat{S}^{x}_{\vec{j}} \hat{S}^{x}_{\vec{j}+\vec{\delta}} + \hat{S}^{y}_{\vec{j}} \cdot \hat{S}^{y}_{\vec{j}+\vec{\delta}} + \Delta \hat{S}^{z}_{\vec{j}} \hat{S}^{z}_{\vec{j}+\vec{\delta}}$

and $\Delta \equiv \frac{U_1}{2J}$

* Note that for $\Delta = 1$ we recover the isotropic Heisenberg Hamiltonian that we discussed in p. (41) when we introduced the idea of super-exchange.

+ GETTING THE XXZ MODEL USING SUPER-EXCHANGE

* In p. (41) we considered the simplified case in which both component of the Bose system had the same tunneling constant. Moreover, we assumed that the interactions between ↑↑, ↑↓ and ↓↓ were characterized by the same coupling constant U.

* If we now relax this assumption we must consider the general Hamiltonian:

$$\hat{H} = -t_\uparrow \sum_{\langle ij \rangle} b^+_{i\uparrow} b_{j\uparrow} - t_\downarrow \sum_{\langle ij \rangle} b^+_{i\downarrow} b_{j\downarrow} + \frac{U_{\uparrow\uparrow}}{2} \sum_i n_{i\uparrow}(n_{i\uparrow}-1)$$

$$+ \frac{U_{\downarrow\downarrow}}{2} \sum_i n_{i\downarrow}(n_{i\downarrow}-1) + U_{\uparrow\downarrow} \sum_i n_{i\uparrow} n_{i\downarrow} - \mu_\uparrow \sum_i n_{i\uparrow} - \mu_\downarrow \sum_i n_{i\downarrow}$$

* If we now repeat our analysis of the Mott regime of p. (41) we recover a <u>spin Hamiltonian of the XXZ form</u> with

$$J \equiv -4 \frac{t_\uparrow t_\downarrow}{U_{\uparrow\downarrow}}$$

and $$J\Delta \equiv \frac{2(t_\uparrow^2 + t_\downarrow^2)}{U_{\uparrow\downarrow}} - \frac{4 t_\uparrow^2}{U_{\uparrow\uparrow}} - 4 \frac{t_\downarrow^2}{U_{\downarrow\downarrow}}$$

Of course, if $t_\uparrow = t_\downarrow = t$ and $U_{\uparrow\uparrow} = U_{\downarrow\downarrow} = U_{\uparrow\downarrow}$, we recover $\Delta = 1$ and $J \equiv -4t^2/U$ as we know already from p. (41).

* Although one may hence obtain the XXZ model also using super-exchange, it is important to note that the energy scales for the case of polar molecules are just t and U_1, which may be significantly larger than the super-exchange energies $\sim t^2/U$.

* Spinor Lattice Gases

* Up to now we have considered at most two-component systems (spin-1/2). A rather rich physics occurs for systems with even more components.
* Spinor gases are formed by atoms with more than one internal (Zeeman) sublevel. These are particularly interesting systems due to the interplay between internal and external degrees of freedom. The breakthrough of spinor gases came with the development of optical (dipole) traps, which allow (contrary to the case of magnetic traps) for the simultaneous confinement of all Zeeman substates.
* Interatomic interactions play a crucial role in the physics of spinor gases. Although dipolar interactions may eventually play an important role, we will focus here on the effect of short-range interactions (s-wave collisions).
* When two particles interact, their spins couple to form a total spin during the collision. Recall that for 2 particles of spin S the total spin (let's call it S_T) ranges from $S_T = 0$ to $2S$ (every 1). Recall that bosons must keep symmetric both in spin and in the motional degree of freedom. Since for s-wave collisions the spatial wavefunction is symmetric, this means that also the spin part must be symmetric. This mean that total spin must be even ($S_T = 0, 2, \ldots$) [for bosons of integer spin]

For fermions (with half spin) the spin wavefunction must be odd, which is fulfilled for even S_T.

[Note: For bosons $S_T + 2S$ must be even, and for fermions odd. Hence for bosons with integer S, and for fermions with half integer we need for both even S_T.]

* Associated with each available value of S_T there's a s-wave scattering length: a_{S_T}, such that the interaction Hamiltonian

is of the form:

$$\hat{H}_{INT} = \frac{1}{2} \sum_{S_T} g_{S_T} \hat{P}_{S_T}$$

where $g_{S_T} = 4\pi\hbar^2 a_{S_T}/m$, and $\hat{P}_{S_T}$ is the projector operator onto a two-particle state with total spin S_T:

$$\hat{P}_{S_T} = \sum_{M=-S_T}^{S_T} \int \hat{A}^{\dagger}_{S_T,M}(\vec{r})\, \hat{A}_{S_T,M}(\vec{r})\, d^3\vec{r}$$

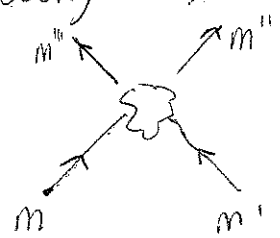
with $\hat{A}^{(\vec{r})}_{S_T,M} = \sum_{m,m'} \underbrace{\langle S_T, M \,|\, m, m' \rangle}_{\text{Clebsch-Gordan coefficients}} \psi_m(\vec{r})\, \psi_{m'}(\vec{r})$

↘ annihilation operator of particles with spin-projection m at site $\vec{r}$. (of course $m = -s \dots s$)

* For example, for $S = 1$ bosons (3-components), the interaction Hamiltonian is of the form:

$$\hat{H}_{INT} = \int d^3\vec{r} \left\{ \left(\frac{g_0+2g_2}{6}\right) \hat{\psi}_0^{\dagger}\hat{\psi}_0^{\dagger}\hat{\psi}_0\hat{\psi}_0 + \left(\frac{g_2}{2}\right)\left(\hat{\psi}_1^{\dagger}\hat{\psi}_1^{\dagger}\hat{\psi}_1\hat{\psi}_1 + \hat{\psi}_{-1}^{\dagger}\hat{\psi}_{-1}^{\dagger}\hat{\psi}_{-1}\hat{\psi}_{-1}\right) \right.$$
$$+ \left(\frac{2g_0+g_2}{3}\right)\hat{\psi}_1^{\dagger}\hat{\psi}_{-1}^{\dagger}\hat{\psi}_{-1}\psi_0 + g_2\left(\hat{\psi}_0^{\dagger}\hat{\psi}_{-1}^{\dagger}\hat{\psi}_{-1}\hat{\psi}_0 + \hat{\psi}_0^{\dagger}\hat{\psi}_1^{\dagger}\hat{\psi}_1\hat{\psi}_0\right)$$
$$\left. + \left(\frac{g_2-g_0}{3}\right)\left(\hat{\psi}_0^{\dagger}\hat{\psi}_0^{\dagger}\hat{\psi}_{-1}\hat{\psi}_1 + \hat{\psi}_1^{\dagger}\hat{\psi}_{-1}^{\dagger}\hat{\psi}_0\hat{\psi}_0\right)\right\}$$

* Note that an important feature of the short-range interactions is that due to their isotropy, they do not modify the spin projection of the pair along the quantization axis.



$\Rightarrow m + m' = m'' + m'''$ (you can easily check this for all terms in the $\hat{H}_{INT}$ above)

• Interestingly this may be achieved in two non-equivalent ways:

• Spin preserving collisions (first 2 lines of $\hat{H}_{INT}$ above): the incoming and the outgoing m's are the same.

• Spin changing collisions (last line of $\hat{H}_{INT}$ above): although $m+m' = m''+m'''$ the individual spin projections are changed: $0,0 \longleftrightarrow +1,-1$

* Note that spin changing collisions have a strength given by $\sim (g_2 - g_0)$, i.e. if $a_2 = a_0$ there're no spin-changing collisions.

* Spin changing collisions play a crucial role in the physics of spinor gases, e.g. in the so-called spinor dynamics recently studied at e.g. the group of C. Klempt at the University of Hannover.

* We will not review in detail the physics of spinor gases, since here we are mostly interested on the physics of spinor lattice gases. As for the spin-less case we may project on the Wannier bans (assuming the single band approximation). Assuming a spin-independent hopping, the Hamiltonian becomes of the form:

$$\hat{H} = -t \sum_{\langle ij \rangle} \sum_{m} \hat{c}^{+}_{im} \hat{c}_{jm} + \frac{1}{2} \sum_{S_T} U_{S_T} \hat{P}_{S_T,i}$$

with $$\hat{P}_{S_T,i} \equiv \sum_{M=-S_T}^{S_T} \hat{A}^{(i)+}_{S_T,M} \hat{A}^{(i)}_{S_T,M}$$

$$\hat{A}^{(i)}_{S_T,M} \equiv \sum_{m,m'} \langle S_T, M | m, m' \rangle \hat{c}_{im} \hat{c}_{im'}$$

* In the following we will be interested in the strong-coupling regime $U_{S_T} \gg t$. We will consider the Mott regime of one-particle per site and derive an effective spin Hamiltonian (as we did in p. 40)

For this purpose we introduce the projector P into states with one particle per site. Let $Q = 1 - P$ the complementary operator.

(Note: $P^2 = P$; $Q^2 = Q$
$PQ = QP = 0$)

Let's consider the eigenvalue problem

$$H|\psi\rangle = E|\psi\rangle \rightarrow H(P+Q)|\psi\rangle = E|\psi\rangle$$

Then $$PH(P+Q)|\psi\rangle = EP|\psi\rangle$$

$$QH(P+Q)|\psi\rangle = EQ|\psi\rangle \rightarrow QHP|\psi\rangle = (E - QHQ)Q|\psi\rangle$$

$$\downarrow$$

$$Q|\psi\rangle = (E - QHQ)^{-1} QHP|\psi\rangle$$

Then: $$PHP|\psi\rangle + PHQ|\psi\rangle = PHP|\psi\rangle + PH(E - QHQ)^{-1} QHP|\psi\rangle$$

- Hence $\underbrace{\left[PHP + PH(E-QHQ)^{-1}QHP\right]}_{\text{effective Hamiltonian}} P|\psi\rangle = EP|\psi\rangle$

(projected on the states with 1 particle per site)

Note that $H = H_{NN} + H_{INT}$:

- $PH_{NN}P = 0$ since H_{NN} moves particles between sites $\bullet\bullet \to \circ\bullet$
- $PH_{INT}P = 0$ since if there's just one particle per site then there's no interaction.

- Hence $PHP = 0$, and we get:

$$\left[PH_{NN}(E-QHQ)^{-1}QH_{NN}P\right]P|\psi\rangle = EP|\psi\rangle$$

(*Note that only H_{NN} connects P with Q and that $QHQ = QH_{INT}Q$)

The effective Hamiltonian is given by super-exchanges between neighboring sites. For neighboring sites i,j we introduce the projectors on the subspace of total spin S_T:

$$P_{ij}(S^T) = \sum_{M=-S_T}^{S_T} |S_T, M\rangle_{ij}\langle S_T, M| \quad \longrightarrow \quad \sum_{S_T=0}^{2S} P_{ij}(S_T) = \mathbb{1}$$

- Then we may introduce this into the effective Hamiltonian

$$\underbrace{\left\{PH_{NN}(E-QHQ)^{-1}QH_{NN}P\left[\sum_{S_T=0}^{2S} P_{ij}(S_T)\right]\right\}}P|\psi\rangle = EP|\psi\rangle$$

$$\left[-\sum_{S_T=0}^{2S} PH_{NN}\frac{1}{U_{S_T}}H_{NN}PP_{ij}(S_T)\right]P|\psi\rangle = EP|\psi\rangle$$

← This is because $E \sim O(t)$ and because for $S_T \to QH_{INT}Q \Rightarrow U_{S_T}$ (all this is correct up to $O(t^2)$)

Then $\underbrace{\left[-\sum_{S_T=0}^{2S}\frac{1}{U_{S_T}}\left(PH_{NN}^2P\right)P_{ij}(S_T)\right]}P|\psi\rangle = EP|\psi\rangle$

H_{eff}

* Let's introduce the permutation operator T_{ij} such that

$$T_{ij}\, |m\rangle_i \otimes |m'\rangle_j = |m'\rangle_i \otimes |m\rangle_j$$

One may easily see that

$$P H^2_{NN} P = 2t^2 \sum_{\langle ij\rangle} \sum_{m,m'} c^+_{im} c_{jm} c^+_{jm'} c_{im'}$$

such that (Bosons: $+$, Fermions: $-$)

$$P H^2_{NN} P\, |m\rangle_i \otimes |m'\rangle_j = 2\,|m\rangle_i \otimes |m'\rangle_j \pm 2\,|m'\rangle_i \otimes |m\rangle_j$$

Hence

$$P H^2_{NN} P = 2t^2 \sum_{\langle ij\rangle} \{ 1_{ij} \pm T_{ij} \}$$

Thus:

$$H_{eff} = \sum_{\langle ij\rangle} \sum_{S_T=0}^{2S} \frac{2t^2}{U_{S_T}} (-1_{ij} \pm T_{ij})\, P_{ij}(S_T)$$

(Fermions: $+$, Bosons: $-$)

One may easily see that

$$T_{ij} = \sum_{S_T=0}^{2S} (-1)^{S_T+2S}\, P_{ij}(S^T) \quad \text{(using parity projectors)}$$

Using that $P_{ij}(S_T)\, P_{ij}(S'_T) = \delta_{S_T, S'_T}\, P_{ij}(S_T)$ we get that

$$(-1_{ij} \pm T_{ij})\, P_{ij}(S^T) = (-1 \mp (-1)^{S_T+2S})\, P_{ij}(S_T)$$

* Hence:

$$\boxed{H_{eff} = \sum_{\langle ij\rangle} \sum_{S_T=0}^{2S} \frac{2t^2}{U_{S_T}} \left[-1 \pm (-1)^{S_T+2S}\right] P_{ij}(S_T)}$$

(Fermions: $+$, Bosons: $-$)

This is the effective super-exchange Hamiltonian.

* We may finally express the projectors $P_{ij}(S_T)$ using spin operators. This is done in the following way:

Let's consider the action of neighbouring operator $\vec{S}_i \cdot \vec{S}_j$ acting over a state with total spin S_T

$$\vec{S}_i \cdot \vec{S}_j \; P_{ij}(S_T)|\psi\rangle = \frac{1}{2}\left[S_T(S_T+1) - 2S(S+1)\right] P_{ij}(S_T)|\psi\rangle$$

It's easy to see that the projectors $P_{ij}(S_T)$ may be expressed as:

$$\boxed{P_{ij}(S_T) = \alpha \sum_{S' \neq S_T} \left[\vec{S}_i \cdot \vec{S}_j - \frac{S'}{2}(S'+1) + S(S+1)\right]}$$

Note: as it should, this operator gives zero when applied to any state with $S \neq S_T$).

The normalization constant α can be easily fixed by acting on the state with total spin S_T:

$$\boxed{\alpha \prod_{S' \neq S_T} \left[\frac{S_T(S_T+1)}{2} + 2S(S+1) - \frac{S'(S'+1)}{2}\right] = 1}$$

- Let's consider now the particular case of spin-1 bosons.

In this case $S_T = 0, 1, 2$ (as we said before $S_T = 1$ won't play a role but this will come naturally from the calculation):

$$\left\{\begin{array}{l} S_T = 2 \longrightarrow (\vec{S}_i \cdot \vec{S}_j)\,|S_T=2\rangle = 1\,|S_T=2\rangle \\ S_T = 1 \longrightarrow (\vec{S}_i \cdot \vec{S}_j)\,|S_T=1\rangle = -1\,|S_T=1\rangle \\ S_T = 0 \longrightarrow (\vec{S}_i \cdot \vec{S}_j)\,|S_T=0\rangle = -2\,|S_T=0\rangle \end{array}\right\}$$

Then:

$$P_{ij}(S_T=2) = \frac{1}{6}(\vec{S}_i \cdot \vec{S}_j + 1)(\vec{S}_i \cdot \vec{S}_j + 2)$$

$$P_{ij}(S_T=1) = \frac{1}{2}(\vec{S}_i \cdot \vec{S}_j - 1)(\vec{S}_i \cdot \vec{S}_j + 2)$$

$$P_{ij}(S_T=0) = \frac{1}{3}(\vec{S}_i \cdot \vec{S}_j - 1)(\vec{S}_i \cdot \vec{S}_j + 1)$$

$S_T = 1$ doesn't contribute due to symmetry.

Hence:

$$\hat{H}_{eff} = \sum_{\langle ij \rangle} \left\{ \frac{2+2}{U_0}\left(-1-(-1)^{0+2}\right) P_{ij}(0) + \frac{2+2}{U_1}\left(-1-(-1)^{1+2}\right) P_{ij}(1) + \frac{2+2}{U_2}\left(-1-(-1)^{2+2}\right) P_{ij}(2) \right\}$$

* Hence: $\hat{H}_{eff} = \sum_{\langle ij \rangle} \left\{ \underbrace{-\frac{4t^2}{U_0}}_{J_0} P_{ij}(0) - \underbrace{\frac{4t^2}{U_2}}_{J_2} P_{ij}(2) \right\}$

$$= \sum_{\langle ij \rangle} \left\{ -J_0 \frac{1}{3} (S_i \cdot S_j - 1)(S_i \cdot S_j + 1) - J_2 \frac{1}{6} (\vec{S}_i \cdot \vec{S}_j + 1)(\vec{S}_i \cdot \vec{S}_j + 2) \right\}$$

$$= \sum_{\langle ij \rangle} \left\{ \left(\frac{J_0 - J_2}{3} \right) - \frac{J_2}{2} (\vec{S}_i \cdot \vec{S}_j) - \left(\frac{J_0}{3} + \frac{J_2}{6} \right) (\vec{S}_i \cdot \vec{S}_j)^2 \right\}$$

· Hence, apart from constant terms we obtain the effective spin Hamiltonian:

$$\boxed{\hat{H}_{eff} = \sum_{\langle ij \rangle} \left[\left(\frac{-J_2}{2} \right) (\vec{S}_i \cdot \vec{S}_j) - \left(\frac{2J_0 + J_2}{6} \right) (\vec{S}_i \cdot \vec{S}_j)^2 \right]}$$ - Bilinear-Biquadratic Hamiltonian

We recover hence an isotropic Heisenberg-like Hamiltonian, but now in addition to the $(\vec{S}_i \cdot \vec{S}_j)$ term we have a quadratic term $(\vec{S}_i \cdot \vec{S}_j)^2$

* For higher spins one gets even higher powers $(\vec{S}_i \cdot \vec{S}_j)^n$ up to $n = 2S$.

---

* To conclude this brief section on spinor lattice gases, let's briefly discuss the effects of the coupling between spin and magnetic field, i.e. the Zeeman effects. At the very low magnetic fields one considers typically in these problems, the Zeeman effect reduces to

· Linear Zeeman effect: This involves a term $H_{LZE} = p \sum_{m,i} m \, c^+_{im} c_{im} = p \sum_i (S^z_i)$

However, since the projection of the spin is conserved, H_{LZE} is a constant energy, and may be hence gauged out.

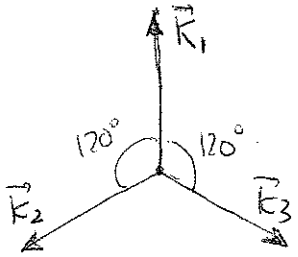
· Quadratic Zeeman effect: $H_{QZE} = q \sum_{m,i} m^2 c^+_{im} c_{im} = q \sum_{m,i} (S^z_i)^2$

This term cannot be gauged out. It actually mimics the so-called single-ion anisotropy in condensed-matter systems.

· Two final notes: (i) Although usually $q \sim B^2$ (B = magn. field), it may be externally mimicked by means of microwave or optical fields; (ii) The QZE may modify the prefactors of the $(\vec{S}_i \cdot \vec{S}_j)^n$ terms.

* LATTICE GEOMETRY

- Up to now we have only considered a simple lattice geometry, namely a square (or cubic lattice). A square lattice is obtained by using 2 pairs of counterpropagating lasers.
- However, proper laser arrangements allow for more complicated (and potentially more interesting) lattice geometries.
- Let's consider for example the case of a triangular lattice, which may be created using 3 laser beams [Becker et al., NJP 12, 065025 (2010)]



- $\vec{k}_1 = k(0, 1)$
- $\vec{k}_2 = k\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
- $\vec{k}_3 = k\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

- We add the field strengths of the individual beams coherently

$$\vec{E}(\vec{r}, t) = \sum_{i=1}^{3} E_{0i}\, \vec{e}_i \cos\left[\vec{k}_i \cdot \vec{r} - \omega t + \phi_i\right]$$

(E_{0i}: amplitude (we assume $E_{0i} = E_0$ for all beams); $\vec{e}_i$: polarization (we assume for all of them linear polarization along $z \rightarrow \vec{e}_i = \vec{e}_z$); ϕ_i: phase)

- The dipole potential created by this laser arrangement is proportional to the field intensity $|\vec{E}(\vec{r}, t)|^2$

$$|\vec{E}(\vec{r}, t)|^2 = \sum_{i,j} |E_0|^2 \{ \cos(\vec{k}_i \cdot \vec{r} + \phi_i) \cos(\vec{k}_j \cdot \vec{r} + \phi_j) \cos^2 \omega t + \sin(\vec{k}_i \cdot \vec{r} + \phi_i) \sin(\vec{k}_j \cdot \vec{r} + \phi_j) \sin^2 \omega t + [\cos(\vec{k}_i \cdot \vec{r} + \phi_i) \sin(\vec{k}_j \cdot \vec{r} + \phi_j) + \sin(\vec{k}_i \cdot \vec{r} + \phi_i) \cos(\vec{k}_j \cdot \vec{r} + \phi_j)] \cos\omega t \sin\omega t \}$$

Averaging over time

$$\overline{|\vec{E}(\vec{r})|^2} = \sum_{i,j} \frac{|E_0|^2}{2} \cos\left[(\vec{k}_i - \vec{k}_j) \cdot \vec{r} + (\phi_i - \phi_j)\right]$$

The relative phases are stabilized (locked) in the experiment. We fix them here to zero.

· Let $\vec{b}_1 \equiv (\vec{k}_2 - \vec{k}_3) = k(-\sqrt{3}, 0)$

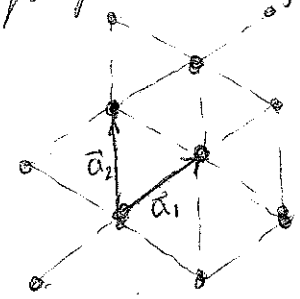
$\vec{b}_2 \equiv (\vec{k}_1 - \vec{k}_3) = \frac{\sqrt{3}}{2} k (1, \sqrt{3})$

Then the potential experienced by the atoms is (apart from constants) of the form:

$$V(\vec{r}) = V_0 \left[\cos(\vec{b}_1 \cdot \vec{r}) + \cos(\vec{b}_2 \cdot \vec{r}) + \cos[(\vec{b}_1 - \vec{b}_2) \cdot \vec{r}] \right]$$

This is a periodic potential. The reciprocal lattice is given by the vectors $\vec{b}_i$ and $\vec{b}_2$. We may then easily evaluate the primitive direct lattice vectors, which fulfill $\vec{a}_i \cdot \vec{b}_j = 2\pi \delta_{i,j}$. A simple calculation gives:

$$\left.\begin{aligned} \vec{a}_1 &= \frac{4\pi}{3k}\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \\ \vec{a}_2 &= \frac{4\pi}{3k}(0, 1) \end{aligned}\right\}$$



$\Rightarrow$ We obtain hence a triangular lattice! (see p. (86)).

* Interestingly one may modify the hoppings in the lattice using the shaking technique discussed in p. (69), such that we get different hopping rates (and eventually with different sign) along $\vec{a}_1$ or $\vec{a}_2$ [Eckhard et al., EPL 89, 10010 (2010)].

One moves the lattice along an elliptical orbit

$$\vec{X}(t) = \Delta X_c \cos\omega t\, \vec{e}_c + \Delta X_s \sin\omega t\, \vec{e}_s$$

The resulting inertial force in the lattice frame is

$$\vec{F}(t) = -m\ddot{\vec{X}} = F_c \cos\omega t\, \vec{e}_c + f_s \sin\omega t\, \vec{e}_s \qquad \text{with } F_{c,s} = m\omega^2 \Delta X_{c,s}$$

· The Hubbard Hamiltonian is then given by:

$$\hat{H}(t) = -\sum_{\langle ij \rangle} t\, \hat{b}_i^\dagger \hat{b}_j + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1) + \sum_i [V_i(t) - \mu] \hat{n}_i$$

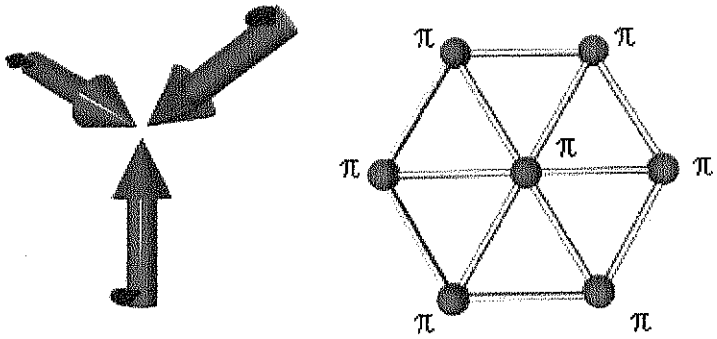
where $V_i(t) = -\vec{F}(t) \cdot \vec{r}_i$

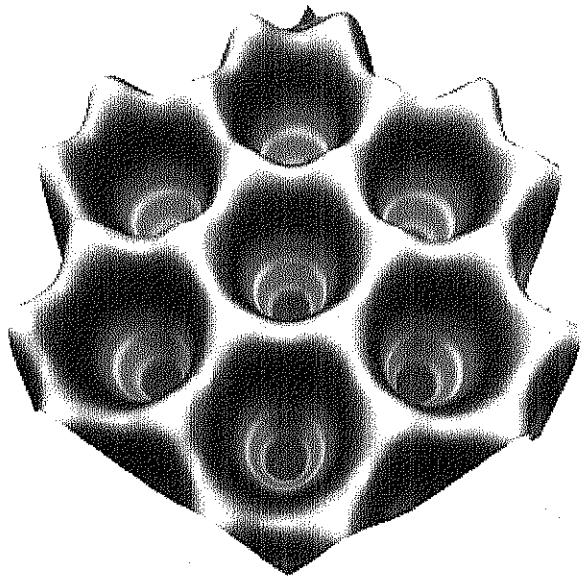
· Using a similar analysis as that of p. (69), this shaking may be translated into an effective hopping rate:

$$t_{ij}^{\text{eff}} = t\, J_0\left[\frac{k_{ij}}{\hbar\omega}\right] \qquad \text{with } k_{ij} = \sqrt{(F_c \vec{e}_c \cdot \vec{r}_{ij})^2 + (f_s \vec{e}_s \cdot \vec{r}_{ij})^2}$$

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

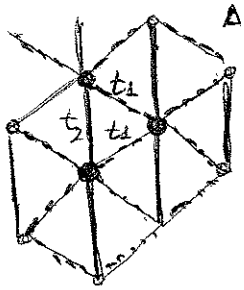
Triangular lattice [Becker et al., NJP 12, 065025 (2010)]





* Let $\vec{e}_c = \vec{e}_y$, $\vec{e}_s = \vec{e}_x$, we get then two different tunneling rates (see figure)

$\Delta x_{c,d} = d$ (lattice constant)



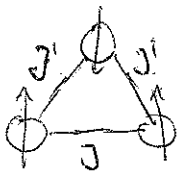
- $t_1 = t\, J_0\left(\frac{K_1}{\hbar\omega}\right)$ with $K_1 \equiv \frac{d}{2}\sqrt{F_c^2 + 3F_s^2}$

- $t_2 = t\, J_0\left(\frac{K_2}{\hbar\omega}\right)$ with $K_2 \equiv d\,|F_c|$

- We may hence obtain any value of the ratio $\frac{t_1}{t_2}$.

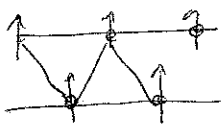
This has been recently employed for studying interesting phases of condensates in triangular lattices in the group of Klaus Sengstock in Hamburg [Struck et al., Science 333, 996 (2011)].

In those experiments the local phase of the condensate at a given lattice site is mapped onto a classical spin vector $\vec{\Phi}$...

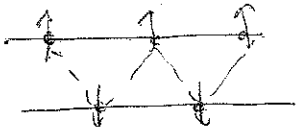


- The coupling $J \equiv t_2$ and $J' \equiv t_1$ may be tuned ferro- or antiferromagnetically, determining the resulting spin configuration (i.e. the condensate phase distribution).

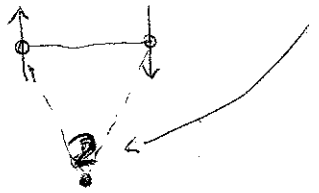
- For example if both couplings are ferro you get



If J' is AF but J is F



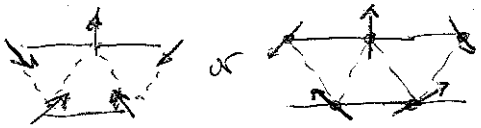
- The funniest situation occur when J is AF, since this leads to frustration i.e. it isn't immediately obvious where the spins should point. Let's see this



It's clear that we can't place here ↑ or ↓ without breaking the F or AF character of at least one bond.

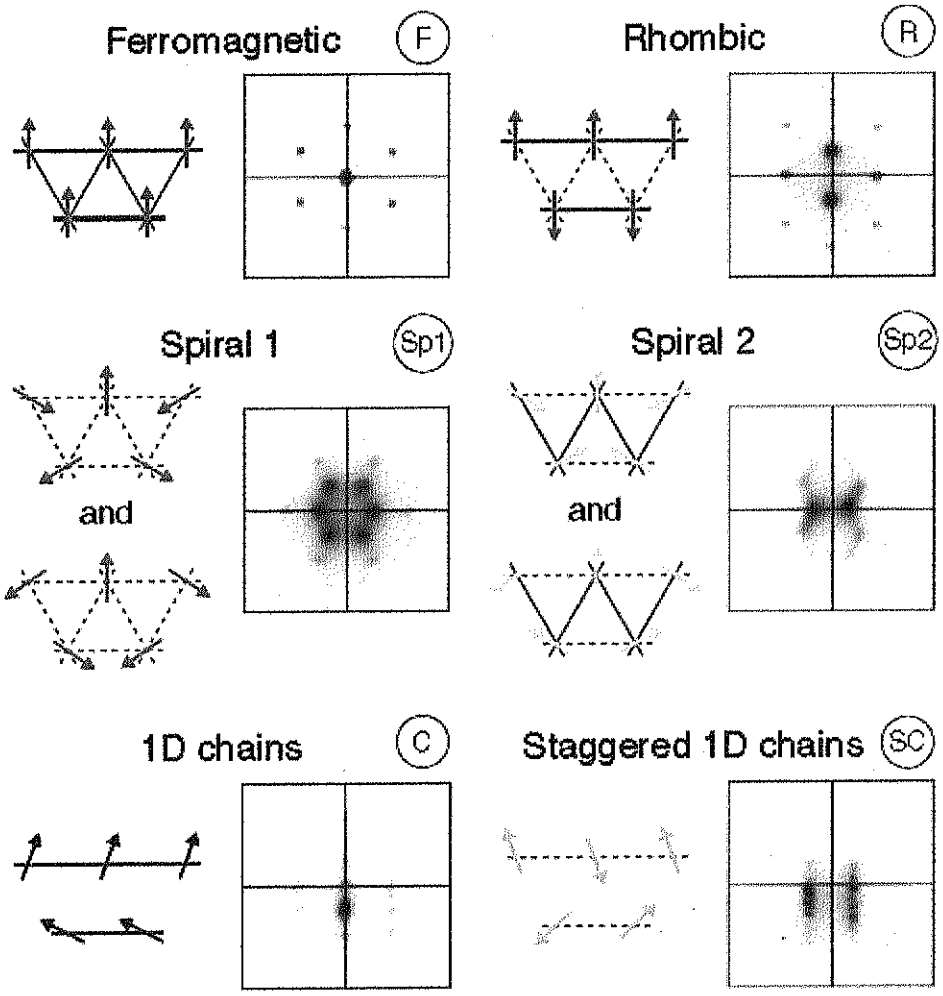
- What one gets is actually an spiral. For J' AF:

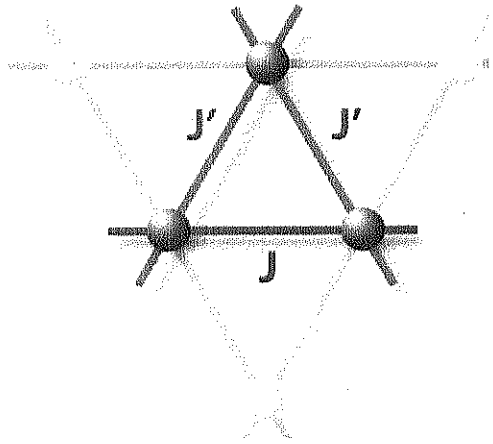
and for J' Ferro:

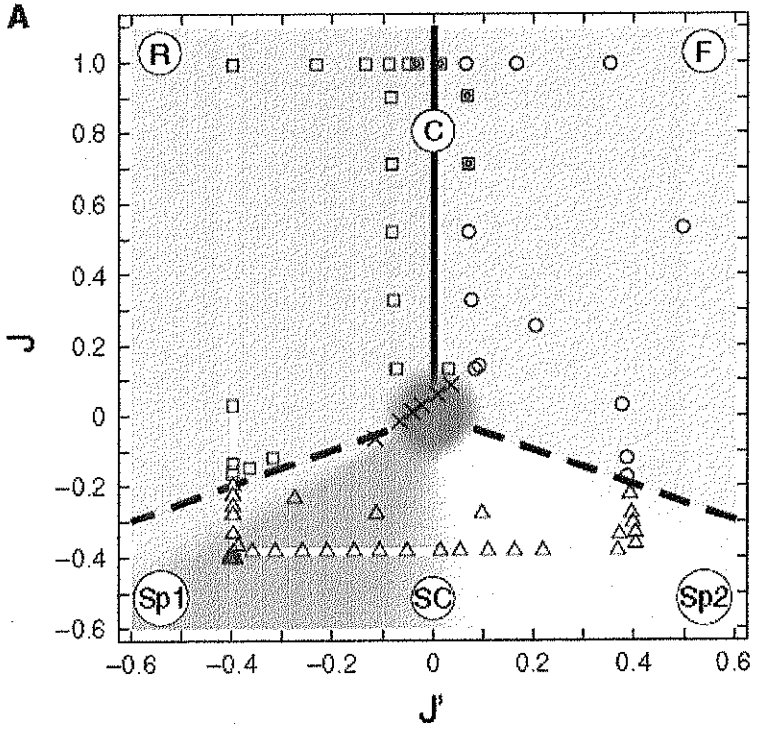


- Frustration is actually one of the most fascinating topics of quantum magnetism, which may be studied with ultra-cold lattice gases.

Triangular lattice [Struck et al., Science 333, 996 (2011)]

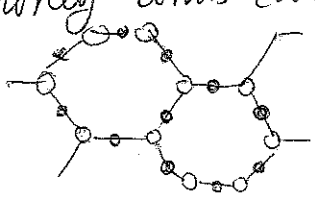






* As mentioned above, 3 plane waves of equal intensity I, equal frequency (ω) and equal $|\vec{k}|$ (and polarization perpendicular to the xy plane) lying in the xy-plane and intersecting at equal angles, produce a triangular lattice of intensity minima. If we employ a <u>blue-detuned</u> lattice (with respect to the atomic transition) then the atoms experience as mentioned above, a triangular lattice of potential minima. [Note: In blue-detuned lattices, intensity minima are potential minima]

* Note from fig (86') that the intensity maxima (of intensity $\frac{9}{2}I$) form a honey-comb lattice. The maxima are separated by a triangular lattice of saddle-points with intensity $4I$ (indicated by •)



. If the lasers are <u>red-detuned</u>, then the potential minima are at the density maxima! → one has a <u>honey-comb lattice</u> with the same token!

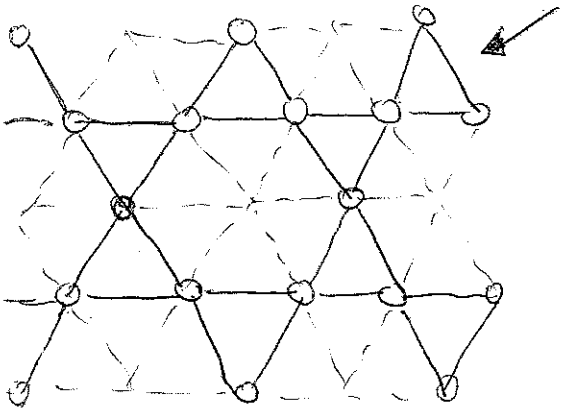
* A combination of a blue-detuned and a red-detuned lattice has been recently employed at the group of D. Stamper-Kurn [Jo et al., arXiv: 1109.1591] to create a so-called <u>Kagomé lattice</u>

They use two lattices (both triangular, created with 3 lasers, but one with $|\vec{k}|$ and the other $|\vec{k}|/2$)

- One with lattice spacing $a/2$ and blue detuned (SW-lattice)
- One with lattice spacing a and red detuned (LW-lattice)

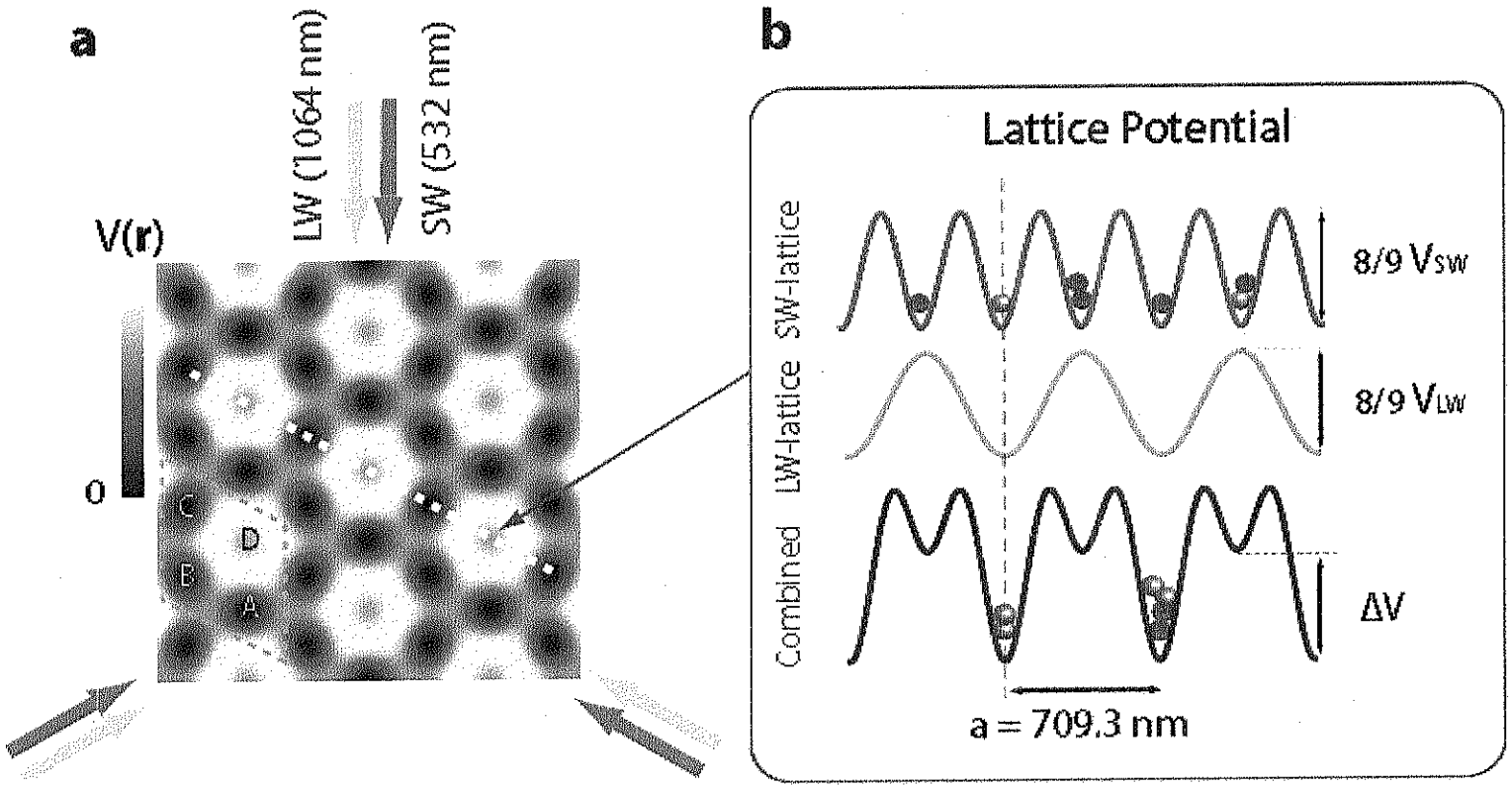
* A unit cell of the LW-lattice has 4 sites of SW-lattice (see figure in p. (88')) A, B, C, D. We may align the lattices such that D coincides with an LW-potential maximum. This lowers the energies of sites A, B, C, giving a barrier $\Delta V = V_D - V_{A,B,C} = \frac{8}{9} V_{LW}$ → depth of the LW lattice

When ΔV increases atoms are excluded from the D sites. The remaining sites form a Kagomé lattice which persists for $V_{LW} > 9 V_{SW}$ → depth of the SW lattice.



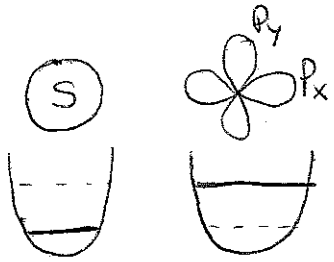
- Kagomé lattices possess interesting properties (including a geometrically induced flat band) Its quantum magnetic properties are actually not fully understood (<u>highly frustrated system</u>)

Kagome lattice [Jo et al., arXiv:1109.1591]



* P-BANDS : ORBITAL SUPERFLUIDITY

* Up to now we have considered that the atoms are always loaded in the lowest band of the lattice (this is the S-band).
* Interestingly one may populate as well higher bands controllably, opening quite fascinating possibilities. One may e.g. populate p-bands.



- The on-site wave functions look then like in the picture.
- This is quite interesting because it opens the possibility of playing with the orbital degree of freedom (i.e. p_x, p_y), whereas for an S-band this degree of freedom was "boring" (always S).

* Particularly interesting in this sense are recent experiments performed at Andreas Hemmerich's group in Hamburg [Wirth et al, Nat. Phys. 7, 147 (2010)].
* They use the laser arrangement depicted in p. (90'). This allows them to create a square lattice composed of two classes of (tube-shaped) lattice sites (A and B). By playing with the relative phase θ between the plane waves that form the lattice (see p. (90')) one can control the relative depth of sites A and B.
- This may be employed to transfer atoms into an upper band (p. (90")). One starts with much deeper B sites (the atoms are in the B minima). One then makes the A sites deeper and then one populates a higher band (it's clear that the A sites are not any more in the ground state). One may then populate the P-band.
- After a short while the atoms condense in the P-band. Note that the P-band has in principle 2 degenerated points at its border (red and blue points in p. (90")). One may control experimentally the relative depth of these minima of the P-band, making them equal (symmetric configuration) or unequal (asymmetric configuration).

* Page (90'''') shows results for an asymmetric case. After a short holding time 2 pronounced peaks are observed in the band mapping corresponding to a condensation in blue (or red) points in p. (90''''). Time-of-flight picture (p. (90'''')) show the appearance of Bragg peaks showing cross-dimensional coherence.

- The symmetric configuration looks however different (p. (90'')) since it shows four peaks. One has there a coherent admixture of both red and blue minima.

Interestingly repulsive interactions favour for a symmetric configuration $p_x \pm i p_y$ complex orbitals, corresponding to a local angular momentum $\pm\hbar$. To maximize intersite hopping, the local phases of adjacent orbitals match at their tunneling junction (see p. (90'')).

One gets then complex BEC wavefunctions [Cai and Wu, PRA 84, 033635 (2011)] of the form:

$$\psi(\vec{r}) = (\psi_{BLUE}(\vec{r}) + e^{\pm i\pi/2}\psi_{RED}(\vec{r}))/\sqrt{2}$$

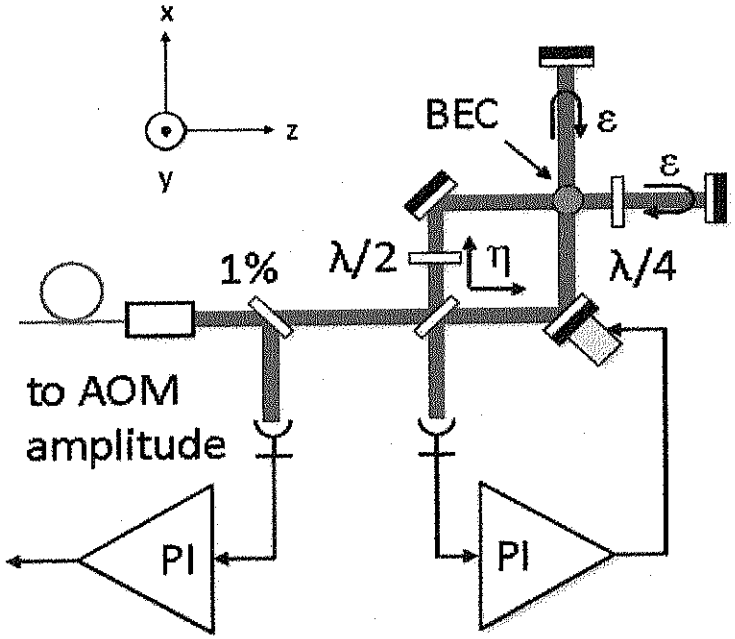
(This complex condensation has been termed orbital superfluidity in Wirth et al.)

* If one breaks the symmetry between the band minima, the energy difference may beat the repulsion energy, and eventually real BECs occur in which either blue or red minima are populated. The orbitals present in that case the shaped form depicted in p. (90''), and one recovers 2 maxima in the band mapping instead of 4. One gets here $p_x \pm p_y$ orbital at sites A (instead of $p_x \pm i p_y$).

- There's hence a competition between band anisotropy (which favors real BEC) and repulsion (which favors complex BEC). As a result there's a second-order phase transition between complex and real BEC for finite values of the asymmetry.

* The possibility of playing with the orbital degree of freedom opens exciting possibilities for strongly-correlated lattice gases as well, as e.g. the possibility of simulating spin-orbital models.

P-bands [Wirth et al., Nature Physics 7, 147 (2010)]



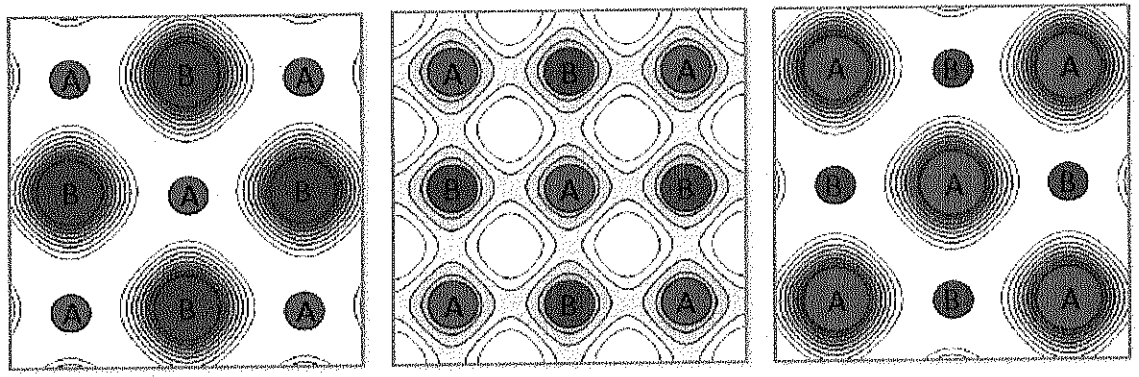
- monochromatic optical lattice with adjustable time-phase difference
- Michelson interferometer with vertical polarization, branches overlap perpendicularly, lattice forms in overlapping region
- Piezo-driven mirror to adjust relative phase difference θ in plane waves
- waveplates rotate polarization by α to compensate backreflection losses ε

$$-\frac{V_0}{4} e^{-\frac{2z^2}{w_0^2}} \Big| \eta \left[\left(\hat{z} \cos(\alpha) + \hat{y} \sin(\alpha) \right) e^{ikx} + \epsilon \hat{z} e^{-ikx} \right] + e^{i\theta} \hat{z} \left(e^{iky} + \epsilon e^{-iky} \right) \Big|^2,$$

For ε=1, α=0:

$$-\frac{V_0}{2} \left(\cos 2k_l x + \eta^2 \cos 2k_l y + 4\eta \cos\theta \cos k_l x \cos k_l y \right)$$

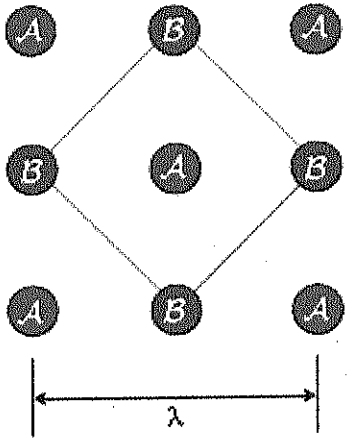
$\theta = 0$ $\theta = \pi/2$ $\theta = \pi$

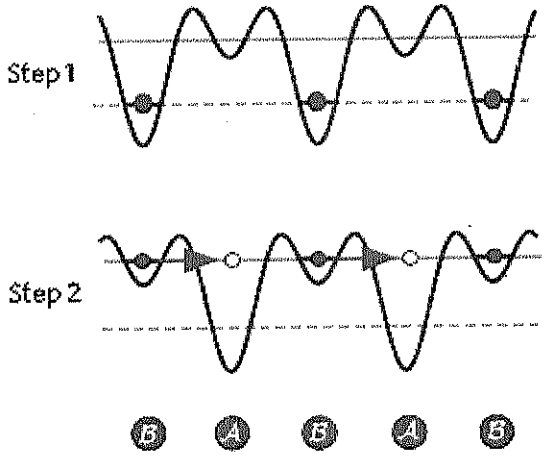


http://photon.physnet.uni-hamburg.de/fileadmin/user_upload/ILP/Hemmerich/miniMOT/Ischgl_2010_Wirth_6.pdf

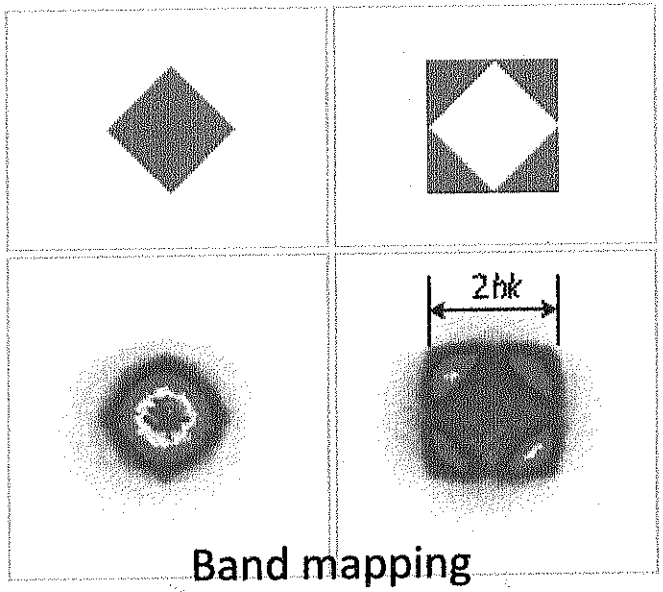
P-bands [Wirth et al., Nature Physics 7, 147 (2010)]

Square lattice of two classes of tube-shaped sites

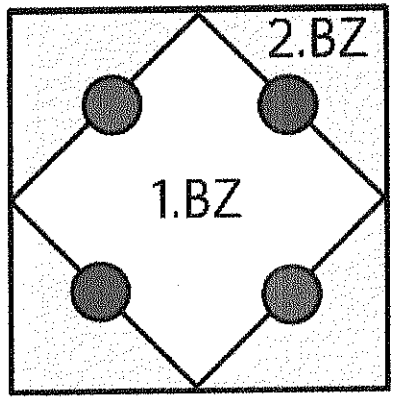




Transfer into p-band

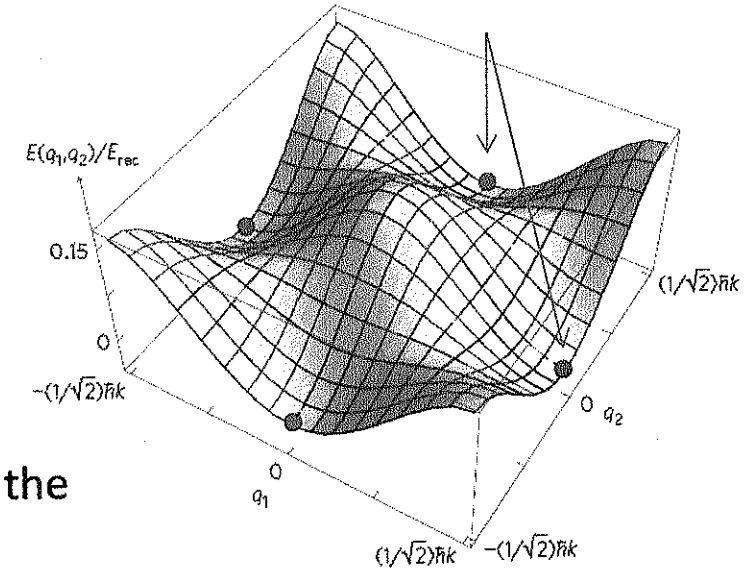


Band mapping



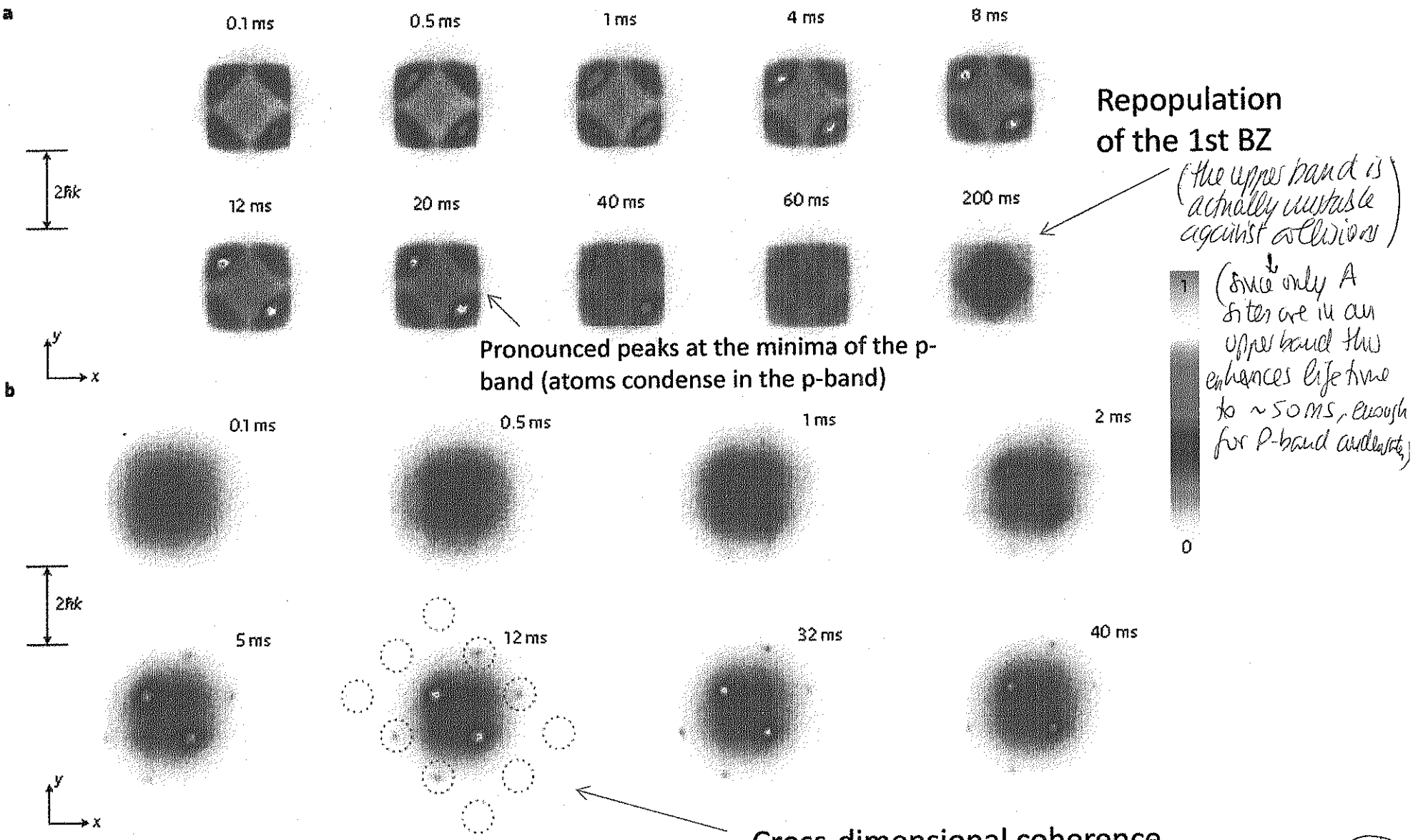
Degenerate points at the edges of the p-band

The degeneracy of the two points may be adjusted experimentally with the value of α



P-bands [Wirth et al., Nature Physics 7, 147 (2010)]

Asymmetric case



Repopulation of the 1st BZ

(the upper band is actually unstable against collisions)

(since only A sites are in an upper band this enhances lifetime to ~50ms, enough for P-band studies)

Pronounced peaks at the minima of the p-band (atoms condense in the p-band)

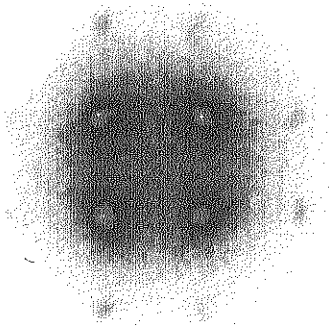
Cross-dimensional coherence

P-bands [Wirth et al., Nature Physics 7, 147 (2010)]

Symmetric vs. asymmetric case

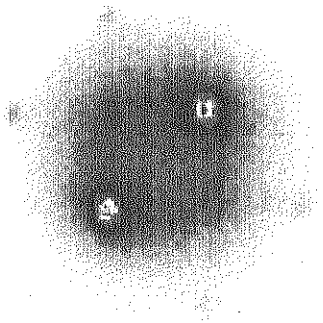
Repulsion favors Px+-iPy orbitals at A sites: Complex BEC !!

Symmetric

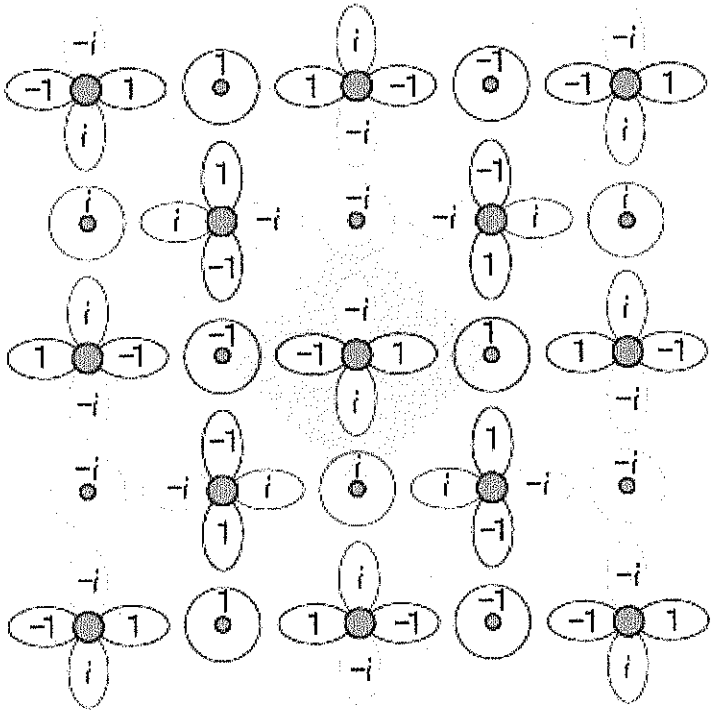


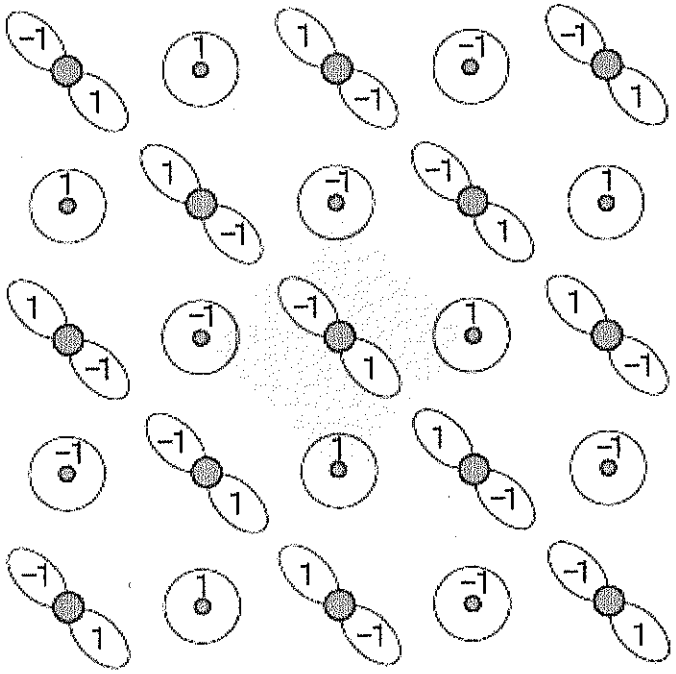
Phase transition controlled by α

Asymmetric



For sufficient asymmetry one recovers a real BEC (px+-py favoured: striped order with zero local angular momentum)





* Effective 3-body hard-core using 3-body losses

* To conclude our (non exhaustive!) discussion about control and manipulation in optical lattices, let us discuss now a very interesting idea introduced by Peter Zoller's group [Daley et al., PRL 102, 040402 (2009)].

* We consider bosons in an optical lattice, which as we have seen already many times may be described under proper conditions by the Bose-Hubbard Hamiltonian:

$$H_{BH} = -t \sum_{\langle ij \rangle} \hat{b}_i^\dagger \hat{b}_j + \frac{U}{2} \sum_i \hat{b}_i^{\dagger 2} \hat{b}_i^2 - \mu \sum_i \hat{b}_i^\dagger \hat{b}_i$$

* In p. (68) we saw that if $U \gg t$ one may enter the so-called hard core regime, in which double occupancy is forbidden ($(b_i^\dagger)^2 = 0$).

* The idea of Zoller's group allows for introducing a 3-body hard core constraint, i.e. $(b_i^\dagger)^3 = 0$, by means of 3-body losses.

* Three-body losses are due to inelastic collisions of 3 atoms, two of which form a deeply bound molecule. Since the binding energy of the molecule is typically much larger than the lattice depth, the products of the inelastic process escape the lattice and are lost.

* We may describe this process by a simple master equation for the density matrix:

$$\dot{\rho} = -i(H_{eff}\rho - \rho H_{eff}^\dagger) + \frac{\gamma_3}{12} \sum_i 2 b_i^3 \rho b_i^{\dagger 3}$$

with $H_{eff} = H_{BH} - i \frac{\gamma_3}{12} \sum_i (b_i^\dagger)^3 b_i^3$

↳ effective Hamiltonian with imaginary (decay) term

$\gamma_3 \equiv$ decay rate

↘ Recycling term coupling sectors of the density matrix differing by 3 atoms (ensures a norm-conserving time evolution)

* We are now interested in the strong-loss limit

$$\gamma_3 \gg t, U$$

which suggests a perturbation theory in $1/\gamma_3$.

* Since γ_3 only affects states with more than 2 particles per site, we define a projector $\mathbb{P}$ onto the subspace with at most 2 atoms per site. Let $\mathbb{Q} = \mathbb{1} - \mathbb{P}$ be the complementary operator.

We may proceed now similarly as we did e.g. in p. (80), to obtain up to 2nd order in perturbation theory the projected Hamiltonian:

$$H_{\text{eff}}^{(\mathbb{P})} \cong \mathbb{P} H_{BH} \mathbb{P} + \mathbb{P} H_{BH} \mathbb{Q} (\mathbb{Q} H_{BH} \mathbb{Q})^{-1} \mathbb{Q} H_{BH} \mathbb{P}$$
$$= \mathbb{P} H_{BH} \mathbb{P} - i \frac{\Gamma}{2} \sum_i \mathbb{P} c_i^\dagger c_i \mathbb{P}$$

with $\Gamma \simeq 12 t^2/\gamma_3$ (effective decay rate) and $c_i = \hat{b}_i^2 \sum_{\langle j \rangle \text{ (neighbor)}} \hat{b}_j/\sqrt{2}$ (jump operator)

We may write the corresponding projected Master equation

$$\dot{\rho}_{\mathbb{P}} = -i \left(H_{\text{eff}}^{(\mathbb{P})} \rho_{\mathbb{P}} - \rho_{\mathbb{P}} H_{\text{eff}}^{(\mathbb{P})\dagger} \right) + \Gamma \sum_i \mathbb{P} \hat{c}_i \rho_{\mathbb{P}} \hat{c}_i^\dagger \mathbb{P}$$

* Let's have a look to the terms of $H_{\text{eff}}^{(\mathbb{P})}$:

* The leading term describes the coherent dynamics of lattice bosons, but with the constraint of a population per site ≤ 2. Hence $\mathbb{P} H_{BH} \mathbb{P}$ has the same form as H_{BH} but now $(\hat{b}_i^\dagger)^3 = 0 \rightarrow$ 3-body hard core!

* The leading correction is imaginary and describes particle number loss. The decay rate $\Gamma \sim t^2/\gamma_3$ is however very small in the limit $\gamma_3 \gg t$. Consequently, over time scales $\tau \sim 1/\Gamma$ one realizes indeed the Bose-Hubbard model with 3-body hard-core constraint.

* This anomalous 3-body hard core opens new possibilities for lattice gases (e.g. for $U<0$ models, where the 3-body constraint prevents on-site collapse even if $|U|/t \gg 1$).

* Indeed, this idea may be placed in the general frame of using dissipation and losses for tailoring lattice models, and also for preparing quantum states of interest.