• Atom Optics

* In the previous lectures we have seen how one can manipulate (and cool) atoms by means of laser fields (also magnetic fields). The manipulation of cold atoms allows for achieving goals analogous to those of conventional optics, as collimating, focusing, reflecting, diffracting and interfering, but with atoms and not with light. The so-called atom-optics deals with these effects.

• In this lecture we assume that
  • The atomic density is sufficiently low → collisions can be neglected.
  • The phase-space density is low-enough → quantum statistics is unimportant.

In future lectures we will see what happens when this is not the case.

If these 2 conditions are fulfilled the physics is actually a single-atom physics (linear atom optics)

The field of atom optics is really very extense, and here we will just deal with some of the basic phenomena, in particular atomic mirrors, atom diffraction and atom interferometry.

Note: other relevant phenomena include
  • Collimation of an atomic beam using radiation pressure
  • Focusing using the dipole force exerted by a Gaussian laser beam or a doughnut mode.
  • Atomic lenses using microfabricated Fresnel lenses
  • Atom holography

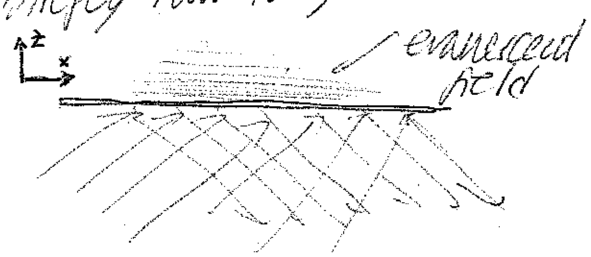
and more. More details can be found e.g. in the book of P. Meystre, "Atom Optics".)

Although some atom optics phenomena may be understood from a quasiclassical picture (where the atom is treated classically), in general atom-optics is based on matter-wave phenomena (the atom behaves as a quantum-mechanical wavepacket). Recall that for a cold atom, Δp is very narrow ($\propto \sqrt{T}$), and hence Δx ($\propto 1/\sqrt{T}$) is very delocalized.

* Atomic Mirrors

* In our discussion on the dipole force (p. 11) we mentioned that a blue-detuned laser induces a repulsive potential. This repulsive potential may be employed to reflect atoms. The idea is actually as simple as that.

* An example of atomic mirrors are the evanescent mirrors. Let's see briefly how they work. Let's consider a prism. A gaussian beam



undergoes total reflection at the dielectric-vacuum interface. When this occurs an evanescent light field is produced at the vacuum side

* Assuming the incident beam in the $\overset{z}{\underset{x}{\llcorner}}$ plane the evanescent wave is of the form

$$\mathcal{E}(x,z) = \mathcal{E}_0 \, e^{-z/k} \, e^{-x^2/w_x^2} \, e^{-y^2/w_y^2} \quad e^{-ik_x x}$$

where w is the gaussian beam waist

$$w_x = w/\cos\theta_i \qquad \text{and } \theta_i \equiv \text{incident angle}$$

$$k = \frac{\lambda}{2\pi} \frac{1}{\sqrt{n^2 \sin\theta_i - 1}} \equiv \text{characteristic decay length} \qquad \begin{pmatrix} n \equiv \text{refraction index} \\ \lambda \equiv \text{wavelength} \end{pmatrix}$$

$$k_x = \frac{2\pi}{\lambda} n \sin\theta_i$$

Forgetting about the x, y dependence (vicinity of $x = y = 0$) one gets a Rabi frequency of the form

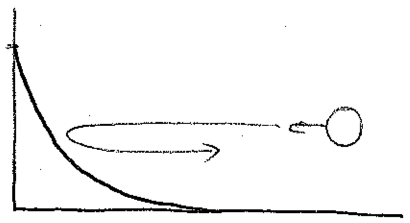
$$\Omega(z) = \Omega(0) \, e^{-z/k}$$

The corresponding dipole force is hence

$$\vec{F}_D = -\hbar\delta \left(\frac{s}{1+s}\right) \frac{\nabla\Omega}{\Omega} \xrightarrow[s \ll 1]{} F_D^{(z)} = \frac{\hbar\delta}{k} \left[\frac{\Omega^2(z)}{\delta^2 + \Gamma^2/4}\right]$$

* If $\delta > 0$ the atom experiences a repulsive force, associated with a repulsive potential of the form (for $\delta \gg \Gamma$):

$$U_D(z) = \frac{\hbar}{4\delta}\Omega(0)^2 e^{-2z/\kappa} = U_0 e^{-2z/\kappa}$$

$\Rightarrow$  This obviously acts as an <u>atomic mirror</u>.

This kind of atomic mirror were first demonstrated in Letokhov's group already in 1987.

(<u>Note</u>: similar atomic mirrors may be generated magnetically, using the evanescent field created at alternating currents or magnets. Here we won't discuss these magnetic mirrors.)

As a final remark concerning atomic reflection, we should point that a potential barrier may be quite simply created by a focused gaussian beam (obviously blue-detuned)

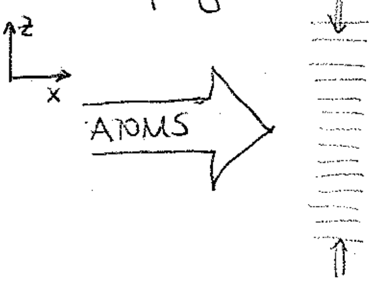
Gaussian potential barrier $\Rightarrow U_0 e^{-x^2/\sigma^2}$

Note that if the kinetic energy $\frac{p^2}{2m} < U_0$ one gets (classically) reflection. However, as we know well from quantum mechanics, is $p^2/2m \sim U_0$ one may get on one side tunneling, i.e. transmission for $p^2/2m < U_0$, and on the other side over-barrier reflection (quantum reflection), i.e. reflection for $p^2/2m > U_0$. Hence the Gaussian laser barrier may be employed also as a semitransparent mirror (i.e. a beamsplitter!).

* Atomic diffraction

* In this section we will discuss the diffraction of atoms by optical gratings (mechanical gratings may be also employed but we won't discuss that). We will see that one has several regimes of diffraction depending on how this is produced.

* A typical atomic diffraction experiment consists of a monoenergetic atomic beam impinging on a standing laser field (of frequency ω)



* In the following (we won't consider here the ~~except~~ Stern-Gerlach regime) we assume that the width (in z) of the impinging beam is large compared with the period of the standing wave (i.e. the atoms probe the whole structure of the laser field grating)

* We will consider as well that the x-velocity of the beam (v_x) is very large (much larger than the recoil) and may be treated classically. In contrast v_z is small and will be treated quantum mechanically.

. In our discussion we forget spontaneous emission effects

. As in several of our previous lectures we write first the atom-laser interaction Hamiltonian (in the rotating frame with frequency ω):

$$\hat{H} = \frac{\hat{P}_z^2}{2M} - \hbar\delta|e\rangle\langle e| + \hbar\Omega_0\cos(kz)\,f(t)\,[|e\rangle\langle g| + |g\rangle\langle e|]$$

$f(t)$ describes the time-profile of the laser$\leftrightarrow$atom interaction. This time just depends on the v_x velocity and the width of the laser region (L). We choose simply
$$f(t) = \Theta(t) - \Theta(t + L/v_x)$$

[Note: $\Theta(t)$ is the Heaviside function]

* Let's consider first the case in which the transverse kinetic energy (given by $\hat{p}_z^2/2m$) is negligible compared with the interaction energy (given by $\hbar\Omega_0$). We will discuss the limit of validity of this approximation later. This diffraction regime is the so-called <u>Raman-Nath regime</u> (also <u>Kapitza-Dirac</u> regime, or <u>thin-grating</u> regime)

$$\hat{H} = -\hbar\delta |e\rangle\langle e| + \hbar\Omega_0 \cos(kz) f(t) \left[|e\rangle\langle g| + |g\rangle\langle e|\right]$$

* Let's use momentum representation ($|p\rangle$). We recall that

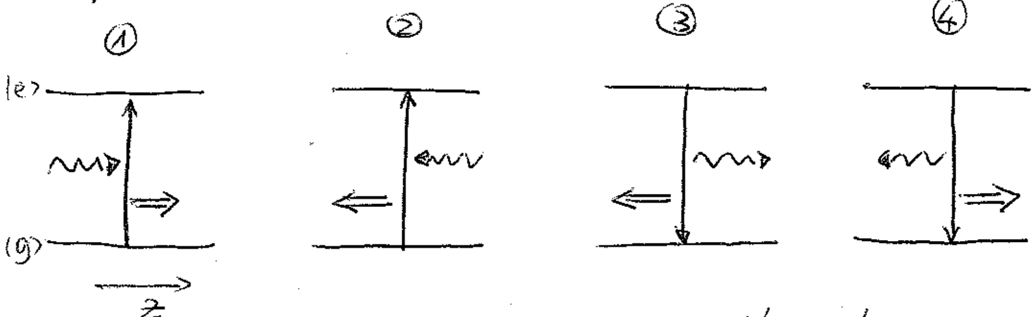
$$\cos(kz)|p\rangle = \frac{1}{2}\left(e^{ikz} + e^{-ikz}\right)|p\rangle = \frac{1}{2}\left[|p+\hbar k\rangle + |p-\hbar k\rangle\right]$$

Let's re-write the Hamiltonian then in the form:

$$\hat{H} = -\hbar\delta \sum_p |e,p\rangle\langle e,p| + \frac{\hbar\Omega_0}{2} f(t) \left\{ \begin{array}{l} \underset{①}{|e,p+\hbar k\rangle\langle g,p|} + \underset{②}{|e,p-\hbar k\rangle\langle g,p|} \\ + \underset{③}{|g,p\rangle\langle e,p+\hbar k|} + \underset{④}{|g,p\rangle\langle e,p-\hbar k|} \end{array} \right\}$$

The physical interpretation is rather transparent

(in the figures: ↑ ≡ ABSORPTION; ↓ ≡ EMISSION; ⤳ ≡ PHOTON; ⟹ ≡ MOMENTUM CHANGE OF THE ATOM)



We observe that the excitation of the atom can result in a momentum kick by $\pm\hbar k$. Similarly the de-excitation can also result is a $\mp\hbar k$ kick. As a result successive absorption + emission processes lead to $p\pm\hbar k, p\pm 2\hbar k, p\pm 3\hbar k, \ldots$

* Let's quantify this effect.

* Let $|\psi(t)\rangle = \sum_p \left[g(p,t)\, |g,p\rangle + e(p,t)\, |e,p\rangle \right]$

We employ the Schrödinger equation $i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$ to obtain:

$$i\hbar \frac{d}{dt} g(p,t) = \frac{\hbar\Omega_0}{2}\left[e(p+\hbar k,t) + e(p-\hbar k,t) \right]$$

$$i\hbar \frac{d}{dt} e(p,t) = \frac{\hbar\Omega_0}{2}\left[g(p+\hbar k,t) + g(p-\hbar k,t) \right] - \hbar\delta\, e(p,t)$$

* Let's focus on the case $\delta = 0$, and the initial condition

$|\psi(t=0)\rangle = |g, p=0\rangle$

From this state we may reach $|e, p=\pm\hbar k\rangle$, $|g, \pm 2\hbar k\rangle$, $|e, \pm 3\hbar k\rangle, \ldots$

This allows us to expand:

$$e(p,t) = \sum_m e_m(t)\, \delta(p - m\hbar k) \qquad (m \text{ odd}), \quad e_m(0) = 0$$

$$g(p,t) = \sum_m g_m(t)\, \delta(p - m\hbar k) \qquad (m \text{ even}), \quad g_m(0) = \delta_{m,0}$$

(Note: in reality there's always an uncertainty in the actual momentum of the photon, and hence the δ must be understood as the idealization of a very sharply peaked function with width $\ll$ recoil).

Then
$$i\hbar \frac{d}{dt} g_m = \frac{\hbar\Omega_0}{2}\left[e_{m+1} + e_{m-1} \right]$$
$$i\hbar \frac{d}{dt} e_m = \frac{\hbar\Omega_0}{2}\left[g_{m+1} + g_{m-1} \right]$$

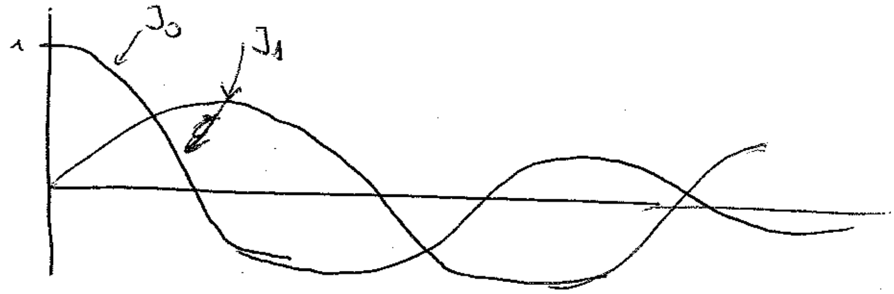
Let $\left.\begin{array}{ll} X_m \equiv g_m & \text{for } m \text{ even} \\ \equiv e_m & \text{for } m \text{ odd} \end{array}\right\} \longrightarrow i\hbar \frac{d}{dt} X_m = \frac{\hbar\Omega_0}{2}\left[X_{m+1} + X_{m-1} \right]$

The solutions of these equations are Bessel functions (of first kind)

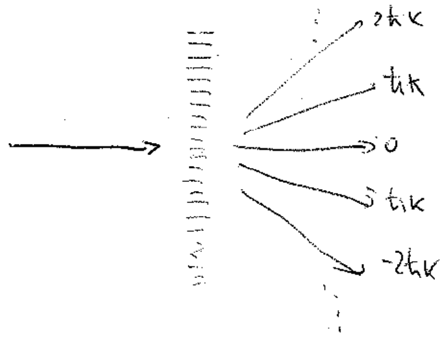
$$X_m(t) = i^m J_m(\Omega_0 t)$$

Hence the probability to find after some time t an atom with momentum $m\hbar k$ is $\boxed{P_m(t) = J_m^2(\Omega_0 t)}$

* I recall you that the Bessel functions $J_m(x)$ have a non-trivial dependence with (quasi-)periodic changes of sign.



* So summarizing in the Raman-Nath regime one gets



$\Longleftarrow$ with a $J_m^2(\Omega_0 t)$ dependence of the different orders.

* As we mentioned above, the Raman-Nath regime is just valid as long as $\hat{P}_z^2/2m$ remains small compared with the interaction energy. Clearly, as more and more scattering orders are excited this condition eventually is violated. Let's estimate this:

- The kinetic energy of the m^{th} order is $m^2 \frac{\hbar^2 k^2}{2m} \equiv m^2 \hbar \omega_{recoil}$
- The interaction energy is $\hbar \Omega_0/2$
- Hence the Raman-Nath condition is $m^2 \omega_{recoil} << \Omega_0/2$
- From the properties of the Bessel functions one may see that for $m>1$ the functions $J_m(u)$ is significantly different than zero only for $m \leq u$, then we can define $m_{max} = \Omega_0 t$, $\longrightarrow m_{max}^2 \omega_{recoil} \leq \Omega_0/2$ implies $\longrightarrow$ $\boxed{t \leq 1/\sqrt{\Omega_0 \omega_{rec}}}$ $\Longleftarrow$ This means that the interaction time must be sufficiently short, either because L is thin or/and v_x is large enough.

* In the Raman-Nath regime, for sufficiently short time, there's a linear increase in the number of scattering orders as a function of time. But this growth is eventually stopped by the effects of the atomic kinetic energy $\hat{p}_z^2/2m$ (which I recall you we neglected in the Raman-Nath approximation).

Physically, this saturation results from the violation of the energy-momentum conservation. Remember ~~that~~:

* For light ~~the~~ dispersion is of the form $E = cp$
* For atom the dispersion is $E = p^2/2m$

Due to this difference it's impossible to conserve both energy and momentum at large scattering orders

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* This discussion brings us to the <u>Bragg regime of atomic diffraction</u> where the effects of this energy-momentum conservation are severe and we can't neglect $\hat{p}_z^2/2m$.

In this case the number of diffracted orders is severely limited and may be as small as 2. $\longrightarrow$ This extreme regime is the <u>Bragg regime</u> (similar ~~to~~ Bragg diffraction in optics)

* This time we employ position representation:

$$|\psi(t)\rangle = \sum_z \left[g(z,t)\,|g,z\rangle + e(z,t)\,|e,z\rangle \right]$$

We apply now the Hamiltonian of page (53) to get

$$i\hbar \frac{\partial}{\partial t} g(z,t) = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2} g(z,t) + \hbar\Omega_0 \cos(Kz)\, e(z,t)$$

$$i\hbar \frac{\partial}{\partial t} e(z,t) = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2} e(z,t) + \hbar\Omega_0 \cos(Kz)\, g(z,t) - \hbar\delta\, e(z,t)$$

* let's assume a very large detuning $|\delta| \gg \Omega_0, \omega_{rec}$ and also that the atoms are initially in their ground state. Then, we can adiabatically eliminate the excited state

(Note: adiabatic elimination amounts basically to second order perturbation theory where the excited state acts as a virtual intermediate state. In our discussion is retrieved by neglecting the $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial z^2}$ derivatives of $e(z,t)$, and then $e(z,t) \simeq \frac{\Omega_0}{\delta}\cos(kz)\, g(z,t)$. Note that due to $|\delta| \gg \Omega_0, \omega_{rec}$ both derivatives of $e(z,t)$ are consistently very small. Check it!)

then:

$$i\hbar \frac{\partial}{\partial t} g(z,t) = \frac{-\hbar^2}{2M}\frac{\partial^2}{\partial z^2} g(z,t) + \frac{\hbar^2 \Omega_0^2}{\delta}\cos^2(kz)\, g(z,t)$$

(Note: the ground state is feeling a potential. This shouldn't be a surprise to us, since it's nothing else as the dipole potential exerted by the laser standing wave!)

* The previous equation is a so-called Mathieu equation.

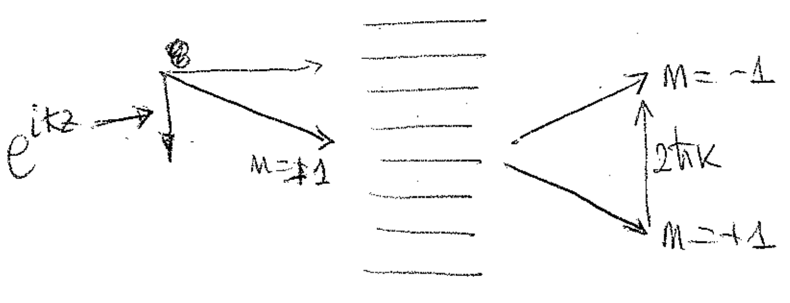
(Note: similar eqs. you find e.g. in solid-state physics as well, when you determine the Bloch states)

* We introduce a Fourier-series expansion:

$$g(z,t) = \sum_n g_n(t)\, e^{inkz}$$

(Note: This is what you typically do when there's a periodic (in this case spatially periodic) driving.)

* We consider the 1st order Bragg scattering $\Rightarrow g_n(0) = \delta_{n,1}$



* Inserting the Fourier-series expansion into the Mathieu equation, we get a set of coupled differential eqs.

$$i\hbar \frac{d}{dt} g_m(t) = \left[m^2 \hbar \omega_{rec} + \frac{\hbar \Omega_0^2}{2\delta} \right] g_m(t) + \frac{\hbar \Omega_0^2}{4\delta} \left(g_{m+2}(t) + g_{m-2}(t) \right)$$

For $m = \pm 1$:

$$i\hbar \frac{d}{dt} g_1(t) = \left[\hbar\omega_{rec} + \frac{\hbar \Omega_0^2}{2\delta} \right] g_1(t) + \frac{\hbar \Omega_0^2}{4\delta} \left(g_3(t) + g_{-1}(t) \right)$$

$$i\hbar \frac{d}{dt} g_{-1}(t) = \left[\hbar\omega_{rec} + \frac{\hbar \Omega_0^2}{2\delta} \right] g_{-1}(t) + \frac{\hbar \Omega_0^2}{4\delta} \left(g_1(t) + g_3(t) \right)$$

Obviously we have an infinite set of coupled eqs.

However, the energy difference between an initial state and a final state separated by m scattering order is

$$\Delta E(m) = \frac{1}{2M} \left((p_i + 2m\hbar k)^2 - p_i^2 \right)$$

$$\overset{p_i = \hbar k}{=} \frac{\hbar^2 k^2}{2M} \left[(1+2m)^2 - 1 \right] = \frac{\hbar^2 k^2}{2M} 4m(m+1)$$

Hence exact energy conservation demands $m = \pm 1$. For $m = \pm 3$, (and long interaction times, i.e. in the Bragg regime) the violation is too large. Hence we can break the hierarchy of equations and get

$$i\hbar \frac{d}{dt} \begin{pmatrix} g_1(t) \\ g_{-1}(t) \end{pmatrix} = \begin{bmatrix} \hbar\omega_{rec} + \frac{\hbar\Omega_0^2}{2\delta} & \frac{\hbar\Omega_0^2}{4\delta} \\ \frac{\hbar\Omega_0^2}{4\delta} & \hbar\omega_{rec} + \frac{\hbar\Omega_0^2}{2\delta} \end{bmatrix} \begin{pmatrix} g_1(t) \\ g_{-1}(t) \end{pmatrix}$$

with the solutions

$$g_1(t) = e^{-i[\omega_{rec} + \Omega_0^2/2\delta]t} \cos[\omega_p t]$$

$$g_{-1}(t) = -i e^{-i(\omega_{rec} + \Omega_0^2/2\delta]t} \sin(\omega_p t]$$

with $\omega_p = \Omega_0^2/4|\delta|$

* Hence Bragg scattering is characterized by (a sort of Rabi-) oscillation between the $n=\pm 1$ scattering orders. These are the so-called Pendellösung oscillations (which are well known in neutron ~~scattering~~ diffraction, and were also observed in cold gases in 1988).

By carefully performing the diffraction we may use Bragg ~~scattering~~ diffraction to construct a beam splitter (see the figure in p. 58).

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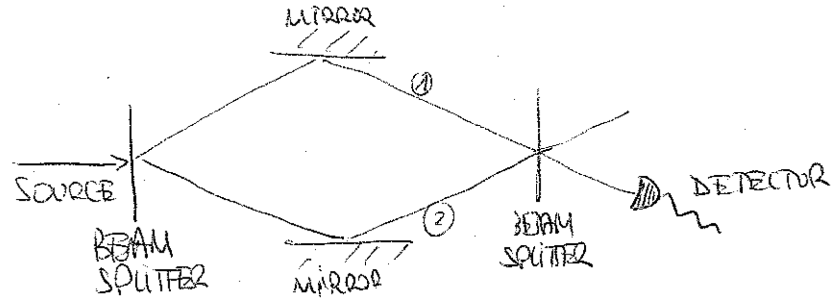
* In the previous discussion we did not consider the effects of the spontaneous emission. As one can expect, the diffraction peaks are smeared out as a result of the random momentum kick imparted on the atom by an spontaneous decay.

One needs hence to avoid spontaneous emission by all means in diffraction experiments, and in particular in the application of matter-wave gratings in atom interferometry. One natural solution is of course to work far off-resonance.

* Atom interferometry

* In the previous lectures we have seen different ways of manipulating atoms with light. These ideas can be employed to construct an atom interferometer.
* An atom interferometer works essentially like a light interferometer. A typical geometry for an interferometer is the so-called Mach-Zehnder geometry:

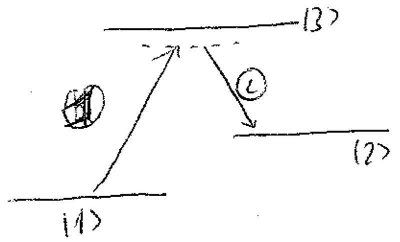


(Interference pattern)

* There are different ways to realize that

* Raman-interferometer

. Let's consider a 3-level atom, and two counterpropagating lasers

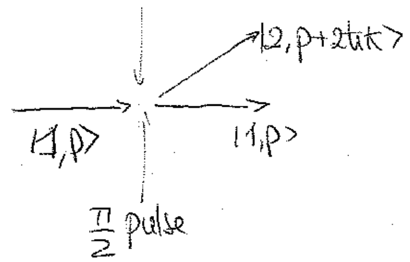


* laser 1: couples $|1\rangle \longleftrightarrow |3\rangle$ (k)
* laser 2: couples $|2\rangle \longleftrightarrow |3\rangle$ $(-k)$

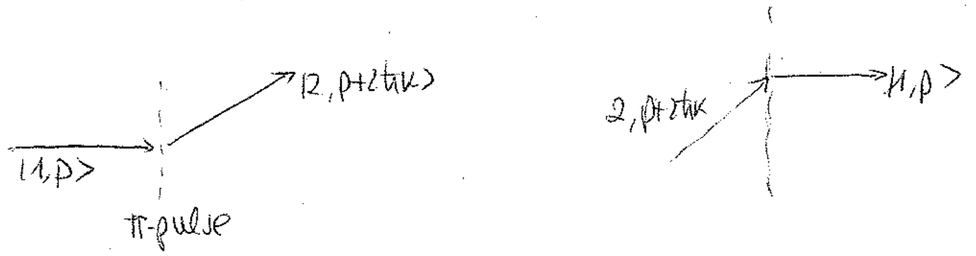
An atom undergoes a Raman transition (i.e. goes up and then down) from $|1\rangle$ to $|2\rangle$. This transition is accompanied by a momentum transfer $2\hbar k$. Hence in this Raman transition we have $|1,p\rangle \longrightarrow |2,p+2\hbar k\rangle$.

. We reduce the problem to a 2 level atom $\{|1\rangle,|2\rangle\}$. Let's suppose that the lasers only act during a ~~[illegible]~~ given pulse time. The 2-level atom is characterized by a Rabi frequency associated to the Raman process. By controlling the pulse duration we may then create different superpositions of $|1\rangle$ and $|2\rangle$.

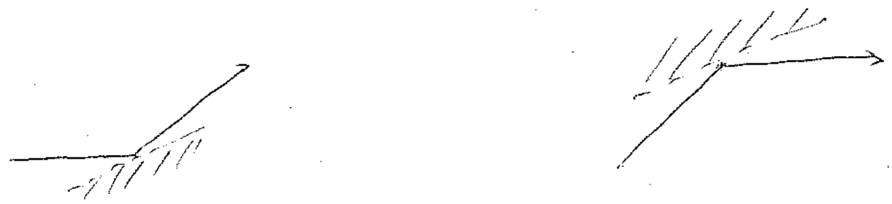
* For example, in a so-called $\pi/2$-pulse (the notation ~~bea~~ motivated actually by the pulse area, but this isn't important here) an initial atom in $|1\rangle$ is transferred into $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$. Due to the momentum transfer this acts like a perfect 50-50 beam-splitter



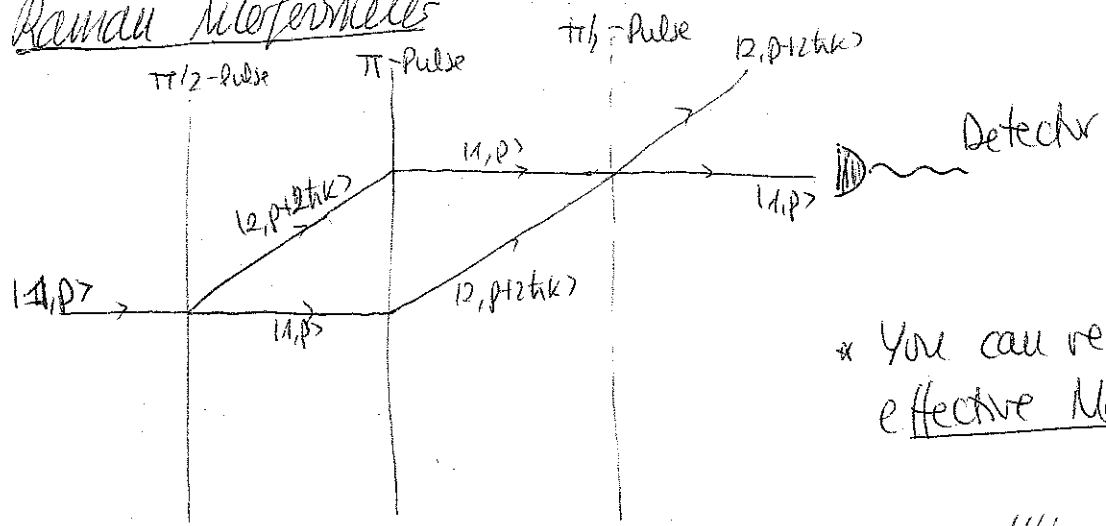
* Other example. A so-called π-pulse swaps $|1\rangle \leftrightarrow |2\rangle$



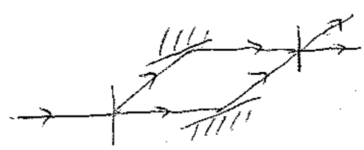
Note that this acts as a mirror for the atom



* We may then combine these ingredients to obtain a Raman interferometer



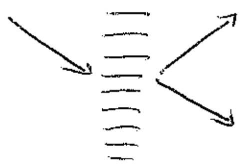
* You can recognize an effective Mach-Zender geometry



* Three-gratings-interferometer

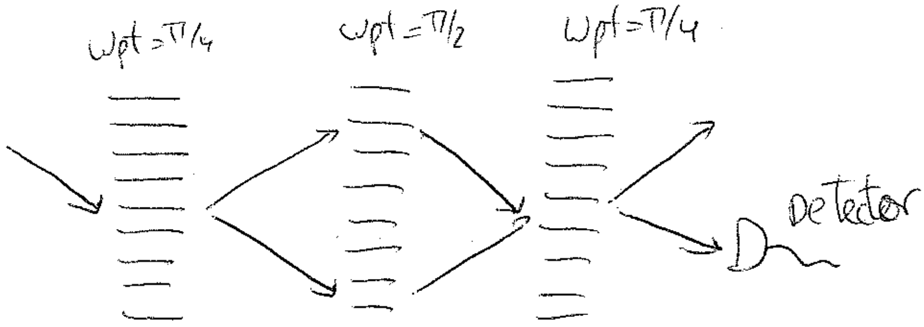
* Alternatively, one may use the previously discussed Bragg diffraction. The effect is very similar as Raman-interferometry.

* Let's consider again our example of p. 59. Remember that depending on the transit time t, and on $\omega_p = \Omega_0^2/4|\delta|$, we had that we could create basically any coherent superposition of the $m=\pm 1$ diffraction orders

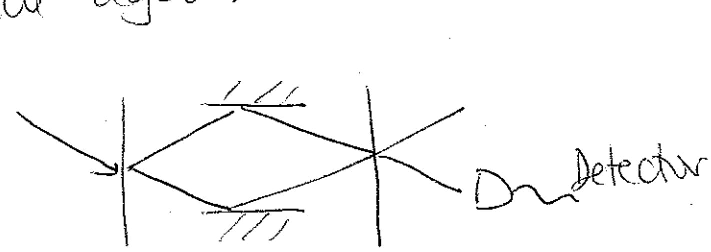
* E.g. $\omega_p t = \pi/4$  $\equiv$ 50-50 Beam splitter

* E.g. $\omega_p t = \pi/2$  $\equiv$ Mirror

* We can then use 3 gratings to build an interferometer



which again acts as



(This interferometer may be actually done also with mechanical gratings)

* Ramsey - Bordé interferometer

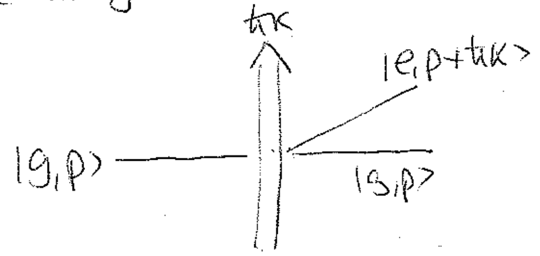
* One may also employ the recoil induced by ~~an~~ photon absorption or emission to build an interferometer. This is the idea of the Ramsey-Bordé interferometer, which employs four running waves.

* Let's consider a 2-level atom $\{|g\rangle, |e\rangle\}$ and a running laser with photon momentum $\hbar k$.

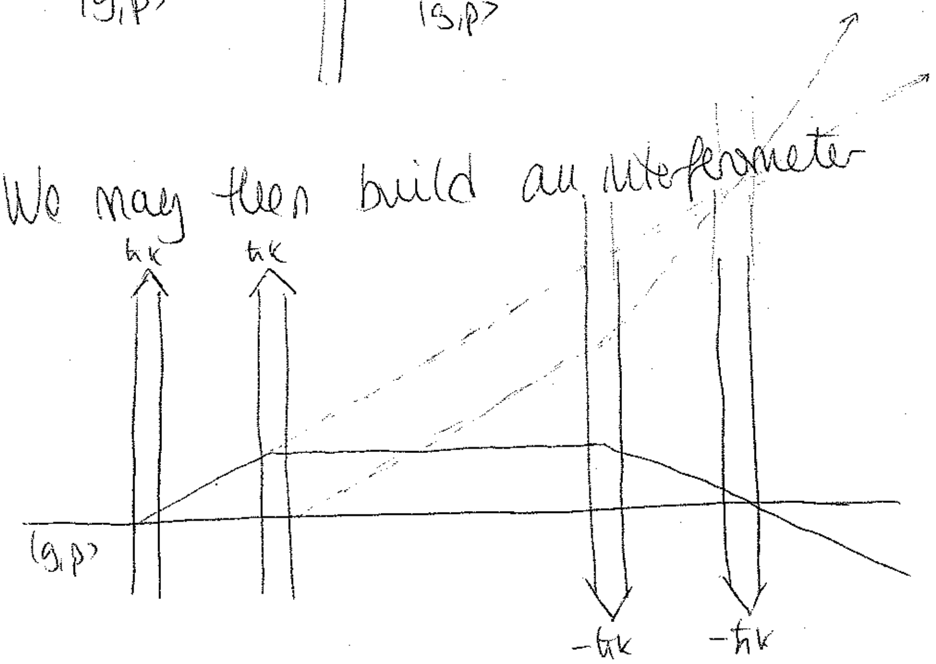
An absorption leads to $|g,p\rangle \longrightarrow |e, p+\hbar k\rangle$

An stimulated emission leads to $|e,p\rangle \longrightarrow |g, p+\hbar k\rangle$

* We may then beam-split:



* We may then build an interferometer



(actually in this arrangement one has 2 possibilities for the interferometer)

* In any case, one way or other, all interferometers may be understood with the scheme of p. 61. The principle of an atom interferometer can be understood by considering the difference between the phases accumulated by the atoms along paths 1 and 2, from source to detector.

* For weak-enough perturbations, the phase difference between the two arms of the interferometer may be evaluated in the WKB approximation (also known as semiclassical approximation), in which we approximate the stationary wave function for an atom of energy E

by $\psi_E(\vec{r}) = A(\vec{r})\, e^{iS(\vec{r})/\hbar}$

where $S(\vec{r})$ is the <u>classical action</u>

Then, the phase accumulated during the propagation through the two arms is $\Delta\Phi = \frac{1}{\hbar}(S_2 - S_1)$

* In the following we will evaluate this action under some general conditions, this will allow us to understand some basic phenomena which may be studied with atom interferometry.

We consider now the dynamics of a single particle under the influence of some general vector and scalar potential $\vec{A}(\vec{r},t)$ and $U(\vec{r})$:

$$H(t) = \frac{(\vec{p} - \vec{A}(\vec{r},t))^2}{2M} + U(\vec{r},t) \Leftarrow \text{Minimal-coupling Hamiltonian.}$$

* The evolution of the particle is hence given by the corresponding Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

* For example:
  * Neutral atom in gravity $\longrightarrow U(\vec{r}) = -Mgz \;, \; \vec{A}(\vec{r}) = 0$
  * Uniform rotation $\vec{\Omega} \longrightarrow U(\vec{r}) = -\frac{M}{2}(\vec{\Omega}\times\vec{r}), \; \vec{A}(\vec{r}) = M(\vec{\Omega}\times\vec{r})$

(Note: a uniform rotation leads 2 a uniform effective magnetic field!)
$\vec{B} = 2\Omega\vec{e}_z M$ for $\vec{\Omega} = \Omega\vec{e}_z$

* We perform first a gauge transformation

$$\psi(\vec{r}) = e^{\frac{i}{\hbar}\int d\vec{r}\cdot\vec{A}(\vec{r})}\,\phi(\vec{r})$$

This transforms

$$H\psi \longrightarrow \left[\frac{p^2}{2m} + U(\vec{r},t)\right]\phi(\vec{r})$$

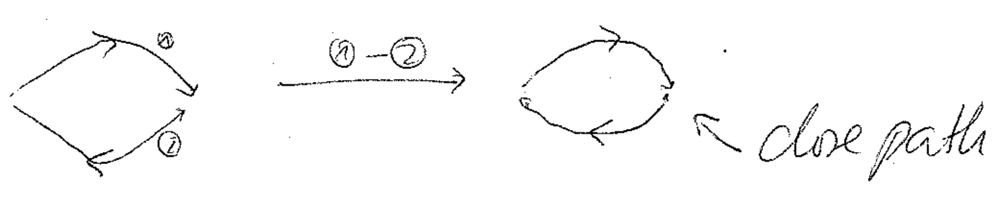
The WKB action is then easily obtained

$$S = \underbrace{\int}_{\text{along the path}} d\vec{r}\,\sqrt{2M(E-U(\vec{r}))}$$

Transforming back to $\psi(\vec{r})$ we obtain the desired phase difference between the paths:

$$\Delta\phi = \frac{1}{\hbar}\oint dr\,\sqrt{2M(E-U(\vec{r}))} + \frac{1}{\hbar}\oint d\vec{r}\cdot\vec{A}(\vec{r})$$

where $\oint$ means the phase when going around the paths



* So the phase difference accumulated by the atoms may be split into 2 contributions:
  * One coming from the scalar potential
    $$\Delta\phi[U] = \frac{1}{\hbar}\oint dr\sqrt{2m(E-U(\bar{r}))}$$
  * One coming from the vector potential
    $$\Delta\phi(\vec{A}) = \frac{1}{\hbar}\oint \bar{A}\cdot d\bar{r} \underset{\text{Stokes' theorem}}{=} \frac{1}{\hbar}\oiint(\bar{\nabla}\times\bar{A})\cdot d\vec{a}$$
    integral over the area of the interferometer

* Let's consider first $\Delta\phi[U]$.

Note that $k(\bar{r}) = \frac{1}{\hbar}\sqrt{2M(E-U(\bar{r}))}$ is the local wave vector.

For a weak potential $U \ll E \rightarrow k(\bar{r}) \simeq k_0 - \frac{1}{2}k_0\frac{U(\bar{r})}{E}$

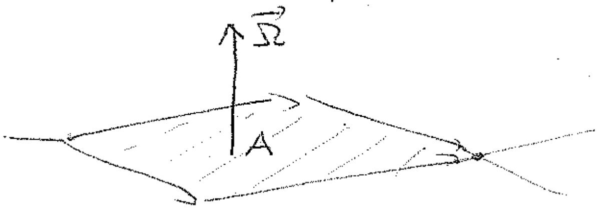
where $k_0 \equiv \frac{\sqrt{2mE}}{\hbar}$. The k_0 term is constant and hence doesn't contribute to the phase difference. Hence

$$\boxed{\Delta\phi[U] \simeq -\frac{1}{\hbar v_0}\oint dr\, U(\bar{r})} \qquad \text{with } v_0 = \hbar k_0/M$$

So the phase shift $\Delta\phi[U]$ is inversely proportional to the atom velocity.

* Let's consider now $\Delta\phi(\vec{A})$. Let's consider specifically the case of rotation $\vec{A} = M(\vec{\Omega}\times\vec{r}) \longrightarrow \bar{\nabla}\times\vec{A} = 2\vec{\Omega}M$

* Let's consider that the rotation axis $\vec{\Omega}$ is perpendicular to the area of the interferometer. Then



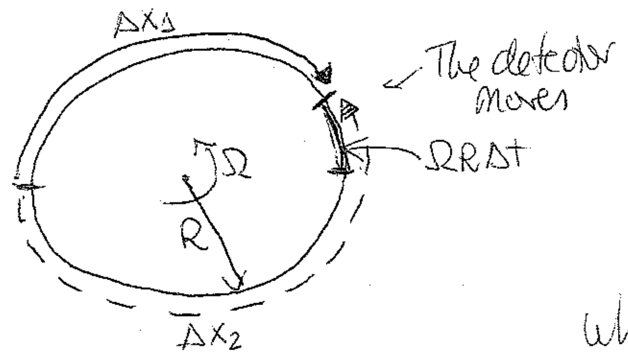
$$\Delta\phi(\vec{A}) = \frac{1}{\hbar}\int 2\vec{\Omega}\, d\vec{a} = \frac{2\Omega}{\hbar} A M$$

where $A \equiv$ area enclosed by the interferometer.

This phase is the so-called <u>Sagnac-effect</u>.

* Contrary to the $\Delta\phi(v)$ this is a geometric shift, only dependent on the topology of the problem.

* The <u>Sagnac effect</u> may be understood very easily. Let's consider a circular interferometer; and let's place ourselves in the rotating frame



* The atoms move with velocity

$$v = P/M = \hbar k/M$$

* The phase difference is

$$\Delta\phi = k(\Delta X_2 - \Delta X_1)$$

where $\Delta X_2 = v\Delta t_2$
$\Delta X_1 = v\Delta t_1$

Note that $v\Delta t_1 = \pi R - \Omega R \Delta t_1$ $\longrightarrow$ $\Delta t_1 = \dfrac{\pi R}{v+\Omega R}$, $\Delta t_2 = \dfrac{\pi R}{v-\Omega R}$

$v\Delta t_2 = \pi R - \Omega R \Delta t_2$

Then

$$\left.\begin{aligned} \Delta t_1 &\underset{v \gg \Omega R}{\simeq} \frac{\pi R}{v}\left(1 - \frac{\Omega}{v}R\right) \\ \Delta t_2 &\simeq \frac{\pi R}{v}\left(1 + \frac{\Omega}{v}R\right) \end{aligned}\right\} \Delta t_2 - \Delta t_1 = \frac{\pi R}{v}\cdot\frac{2\Omega}{v}R$$

Hence

$$\Delta\phi = k\cdot\cancel{v}\cdot\frac{\pi R^2}{\cancel{v}}\cdot\frac{2\Omega}{v} = 2(\pi R^2)\Omega\frac{k}{v} = 2A\Omega\frac{M}{\hbar}$$

as we had above!

* A similar effect is well known in optics.

In optics
$$\Delta\phi = \frac{4\pi A\Omega}{c\lambda} = 2\frac{A\Omega}{\hbar}\left(\frac{\hbar\omega}{c^2}\right)$$

Then $\frac{\hbar\omega}{c^2}$ works as an "effective mass" for the optical interferometer.

* Then (everything being equal!) a matter-wave rotation sensor is more sensitive than an optical sensor by a factor $Mc^2/\hbar\omega$, this factor is <u>huge</u>, typically $\sim 10^{11}$.

Of course not everything is equal (e.g. the area is much larger in optical sensors) but nevertheless this promised sensitivity makes atomic rotation sensors a very active research field nowadays (e.g. the group of E. Rasel here in Hannover).

* Matter-wave interferometry has indeed many practical applications
  * gravimetry
  * Precision tests of fundamental physical theories
  * Measurement of e.g. the gravitational constant
  * Measurement of the constancy of fundamental constants

etc. It's a very active field and a detailed discussion would take us a full semester.

* We will leave here atom optics, and move to the regime of quantum degeneracy.