• COHERENCE

• An important consequence of BEC is the occurrence of coherent effects
associated with the phase of the order parameter. In this lecture
we will have a look to some interesting phenomena (and ideas) related
to the concept of coherence.

DOUBLE-SLIT EXPERIMENT

• As we already discussed in p. 80, the concept of BEC is intimately
related to the idea of off-diagonal long-range order (ODLRO). Remember
that for

$$T > T_c \longrightarrow \rho(s) \cong n\, e^{-\pi s^2/\lambda_T^2}$$

$$T < T_c \longrightarrow \rho(s) \cong n_0 + \frac{1}{(2\pi)^3 \lambda_T^3}\,\frac{1}{s}$$

where $\rho(\vec{r}-\vec{r}\,') = \langle \psi^+(\vec{r})\,\psi(\vec{r}\,')\rangle$, and λ_T is, as always, the thermal
de Broglie wavelength, and n_0 is the condensate density.

• Actually, full BEC implies

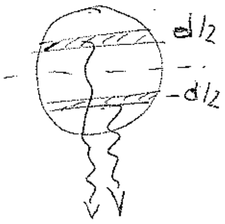
$$\rho(\vec{r}-\vec{r}\,') = \psi_0^*(\vec{r})\,\psi_0(\vec{r}\,') = \sqrt{n(\vec{r})}\,\sqrt{n(\vec{r}\,')}$$

You may recognize here the definition of first-order coherence which
is pretty familiar in optics.

Following this analogy, we may define the coherence length as the
distance s at which the one-body density matrix $\rho(s)$ becomes
negligible. It's clear that for a pure BEC the coherence length is of
the order of the sample size. Notice that below T_c the coherence
length will become typically much smaller than the system size.

• In order to probe this one needs interferometric experiments.
An important experiment in this sense is the analogue of the
double slit experiment (**Note**: Young's double slit is also the typical way
of studying 1st order coherence in optics)

* The double slit for trapped atoms is created using rf fields with 2 frequency components acting on a BEC trapped magnetically (remember our discussion on rf-knives in p. 48). This generates two "leaks" in the BEC (two slits) at different heights from which atoms escape and propagate downwards in gravity. The correlation function (the 1-body density matrix) is then revealed by the interference pattern of the two propagating matter waves.



* The initial phase of each wave is fixed by the value of the field operator $\hat{\psi}$ at the position $\pm d/2$ of each slit (see figure), and hence the visibility of the interference pattern is proportional to

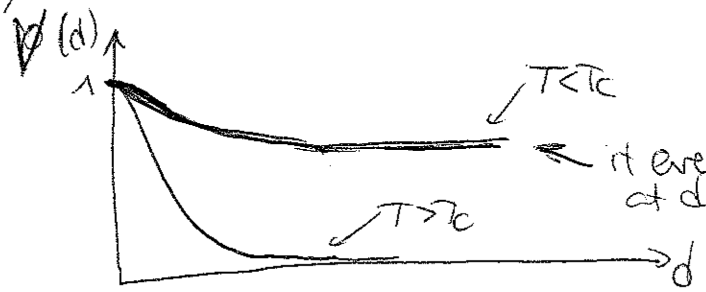
$$\langle \hat{\psi}^{+}(z+d/2)\,\hat{\psi}(z-d/2)\rangle = \rho\,(z+d/2,\, z-d/2)$$

* let's assume d much smaller than the BEC size in the z-direction (the density, i.e. $n(z) \equiv \rho(z,z)$, is uniform between the slits).

We may then define the visibility factor

$$V = \frac{\rho\,(z+d/2,\, z-d/2)}{n(z)}$$

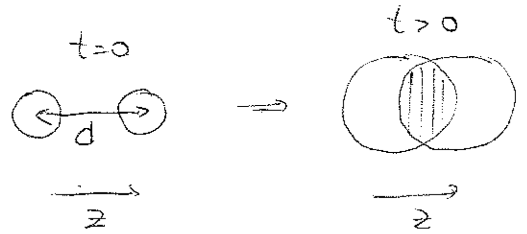
* One may then analyze $V(d)$ which gives the information about $\rho(d)$. The experimental results are in striking agreement with the expected behavior (Bloch et al., Nature 2000).



← it eventually tends to zero at distances of the order of the BEC size.

• INTERFERENCE BETWEEN TWO CONDENSATES

• According to the Gross-Pitaevskii theory, the BECs behave like classical matter waves (with an amplitude and a phase) and hence they may interfere with each other. The simplest example of interference is produced by two initially separated BECs, which after switching-off the traps expand, and eventually overlap (and interfere) (Ketterle's group, 1997)



• At $t=0$ the order parameter is described by the linear combination:

$$\boxed{\Psi(\vec{r}) = \Psi_a(\vec{r}) + e^{i\Phi}\Psi_b(\vec{r})}$$

• The initial wave function of the 2 BECs ($\Psi_a(\vec{r})$ and $\Psi_b(\vec{r})$) are centered at a relative distance d, such that they don't overlap at $t=0$. Φ is the relative phase between the 2 condensates.

• When the two condensates overlap the total density $n(\vec{r},t) = |\Psi(\vec{r},t)|^2$ becomes:

$$n(\vec{r},t) = n_a(\vec{r},t) + n_b(\vec{r},t) + 2\sqrt{n_a(\vec{r},t)\,n_b(\vec{r},t)}\,\cos\left[\Phi + \frac{md}{\hbar t}d\right]$$

Note: remember our discussion about the BEC expansion (p. 156). Recall that for a sufficiently long time the self similar parameter goes as $b(t) \simeq \omega t$. Recall also (p. 155) that the phase goes as $\frac{1}{2}\frac{m}{\hbar}\alpha r^2$, where $\alpha = \dot{b}/b = 1/t$, then the phase goes as $S(\vec{r}) = \frac{m r^2}{2\hbar t}$. Then $S(x,y,z+d/2) - S(x,y,z-d/2) = \frac{m}{2\hbar t}\left[(z+d/2)^2 - (z-d/2)^2\right] = \frac{md}{\hbar t}z$, which is exactly the extra phase appearing above in the cosine factor).

• The interference pattern is characterized by an inter-fringe separation $\lambda = \frac{2\pi\hbar}{md}t$, while the fringe position is fixed by the initial value of the relative phase Φ. Notice also that the cross-term goes as $\sqrt{n_a n_b}$ and hence it's zero at $t=0$ (no initial overlapping) but it becomes progressively more important later during the overlapping during the expansion.

* In the previous discussion we have had a look to what happens if at $t=0$ the condensates are in the initial coherent superposition $\psi(\vec{r}) = \psi_a(\vec{r}) + e^{i\phi}\psi_b(\vec{r})$. Let's see now what would happen if this weren't the case, i.e. if <u>the two samples were independent with an undetermined value of the relative phase.</u>

. Let $\hat{a}^+$ and $\hat{b}^+$ be the creation operators corresponding to the single-particle states $\phi_{a,b} = \psi_{a,b}/\sqrt{N_{a,b}}$

To a first approximation, we may describe the independent configuration as a Fock state of the form:

$$|K\rangle = \frac{1}{\sqrt{(\frac{N}{2}+K)!\,(\frac{N}{2}-K)!}}\,(\hat{a}^+)^{\frac{N}{2}+K}\,(\hat{b}^+)^{\frac{N}{2}-K}\,|VAC\rangle$$

where $|vac\rangle$ is the state vacuum of particles. Note that $|K\rangle$ is an exact eigenstate of $\hat{N}_a = \hat{a}^+\hat{a}$ and $\hat{N}_b = \hat{b}^+\hat{b}$

$$\hat{N}_a|K\rangle = N_a|K\rangle = (N/2 + K)|K\rangle \qquad (N = \text{total number of atoms})$$

$$\hat{N}_b|K\rangle = N_b|K\rangle = (N/2 - K)|K\rangle$$

(<u>Note</u>: This state $|K\rangle$ describes a so-called <u>fragmented condensation</u>, where more than one state (in this case 2, i.e. ϕ_a an ϕ_b) are macroscopically occupied.)

. The coherent configuration discussed in the previous page is on the contrary a state of the form:

$$|\Phi\rangle = \frac{1}{\sqrt{N!\,2^N}}\,(\hat{a}^+ + e^{-i\phi}\hat{b}^+)^N\,|VAC\rangle$$

i.e. all N atoms are in the same single-particle state $(\phi_a + e^{i\phi}\phi_b)/\sqrt{2}$

Note that $|\Phi\rangle$ is NOT an eigenstate of $\hat{N}_{a,b}$, although $\langle\Phi|N_{a,b}|\Phi\rangle = N/2$. Actually it's easy to see that

$$|\Phi\rangle \overset{\substack{\text{binomial formula}\\ \downarrow}}{=} \frac{1}{\sqrt{N!\,2^N}} \sum_{k=-N/2}^{N/2} \frac{N!}{\left(\frac{N}{2}+k\right)!\left(\frac{N}{2}-k\right)!} (\hat{a}^\dagger)^{\frac{N}{2}+k} (\hat{b}^\dagger)^{N/2-k} e^{-i\phi\left(\frac{N}{2}-k\right)}$$

$$= e^{-i\phi\frac{N}{2}} \sum_{k=-N/2}^{N/2} C(k)\, e^{i\Phi k} |k\rangle$$

where $C(k) \equiv \sqrt{\dfrac{N!}{2^N\left(\frac{N}{2}+k\right)!\left(\frac{N}{2}-k\right)!}} \underset{\text{Stirling formula}}{\sim} e^{-k^2/N}$

Then $\langle\phi|(\hat{N}_a-\hat{N}_b)^2|\phi\rangle \sim \sum_k k^2 e^{-k^2/N} = N$

$$\left(\langle\phi|(\hat{N}_a-\hat{N}_b)|\phi\rangle\right)^2 \sim \left(\sum_k k\, e^{-k^2/N}\right)^2 = 0$$

Then the variance $\Delta(\hat{N}_a-\hat{N}_b) \simeq \sqrt{N}$

* Let's suppose that at $t=0$ the system is in $|k=0\rangle$. ~~We~~ neglect interaction between the BECs (which is quite o.k. due to the large decrease of density during the expansion).

Note that

$$\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{i\frac{N}{2}\Phi} |\phi\rangle = \sum_k C(k) \underbrace{\left[\int_{-\pi}^{\pi} e^{ik\Phi} \frac{d\phi}{2\pi}\right]}_{\delta_{k,0}} |k\rangle = C(0)\,|k=0\rangle$$

Hence $|k=0\rangle = C(0)^{-1} \displaystyle\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{i\frac{N}{2}\Phi} |\phi\rangle$

Hence the measurement of the interference fringes of the two samples after the expansion, which gives access to the value of Φ, corresponds to a typical problem of reduction of the wave packet. Hence, the measurement will provide a phase $\Phi \in (-\pi,\pi)$ with equal probability, and after the measurement the system is projected into the corresponding state $|\Phi\rangle$.

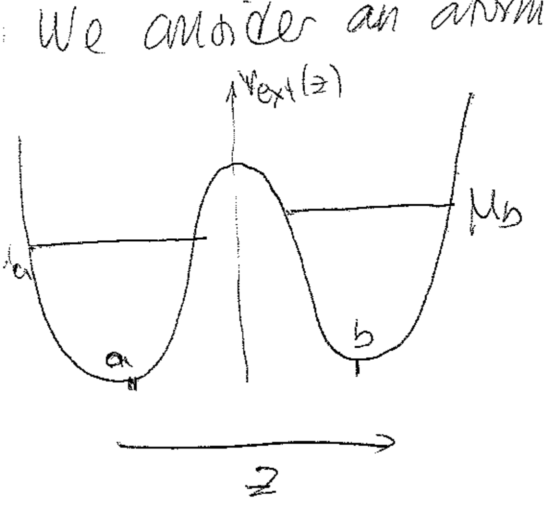
- Hence each measurement of the interference pattern starting from a state $|k\rangle$ (for other k values is the same) will then provide an interference pattern in the (expanded) density as in p. 167. However the relative phase Φ is chosen (randomly) at the projection, and hence will be different for different measurements. Hence in average the interference pattern is washed-out.

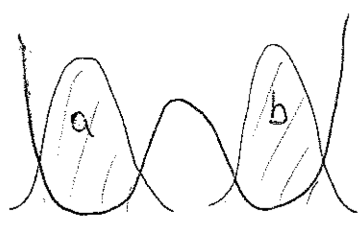
- The situation is crucially different if the initial configuration corresponds to a $|\Phi\rangle$ state. In this case the value of Φ is always the same, and hence the interference fringes are kept after averaging over many experiments!

- JOSEPHSON EFFECT (MEAN-FIELD ANALYSIS)

- The Josephson effect is an important quantum phenomenon that consists of a coherent flow of particles which tunnel through a barrier in the presence of a chemical potential gradient.
It has been observed in superconductors, superfluid Helium, and very recently (Oberthaler's group) in atomic BECs. /2005
- In the following we will discuss the effect in the mean field formalism (beyond mean-field effects will be discussed later).

- We consider an atomic gas in a 2-well potential as in the figure.





- We assume the barrier high enough such that the overlap is small

* For a given number of particles $N_{a,b}$ we may calculate the corresponding wave function $\psi_{a,b}(\vec{r}, N_{a,b})$ at each well (we recall that the overlapping is very small). Note that $\int |\psi_{a,b}|^2 d^3r = N_{a,b}$ and $N = N_a + N_b$. Note also that $\psi_{a,b}$ are a function of $N_{a,b}$ due to interactions.

* The ground state is given by the symmetric combination:

$$\psi(\vec{r},t) = \left[\psi_a(\vec{r}, N/2) + \psi_b(\vec{r}, N/2)\right] e^{-i\mu t/\hbar} \quad \begin{pmatrix}\text{we assume} \\ \text{here } \mu_a = \mu_b = \mu\end{pmatrix}$$

* We may also write non-stationary solutions of the time dependent GPE describing the exchange of particles between the two wells.

$$\psi(\vec{r},t) = \psi_a(\vec{r}, N_a) e^{iS_a} + \psi_b(\vec{r}, N_b) e^{iS_b}$$

where $N_{a,b}$ and $S_{a,b}$ are time dependent (but $N = N_a + N_b$ is fixed)

* We may then calculate the current density (from left (a) to right (b) in z)

$$j = \frac{-i\hbar}{2m}\left(\psi^*(\partial_z\psi) - (\partial_z\psi^*)\psi\right) =$$

$$= \frac{-i\hbar}{2m} \cdot 2i\, \mathrm{Im}\left[\psi_a \partial_z \psi_b\, e^{-i(S_a - S_b)} + \psi_b \partial_z \psi_a\, e^{i(S_a - S_b)}\right]$$

$$= \frac{-\hbar}{m}\left(\psi_a \partial_z \psi_b - \psi_b \partial_z \psi_a\right) \sin\Phi \quad \text{with } \Phi = S_a - S_b$$

Due to the continuity equation $\frac{\partial \rho}{\partial t} = -\frac{\partial j}{\partial z}$, we may easily obtain that

$$\frac{\partial N_a}{\partial t} = I_J \sin\Phi = \frac{-\partial N_b}{\partial t}$$

where $I_J \equiv \frac{\hbar}{m}\int dx \int dy \left[\psi_a \frac{\partial}{\partial z}\psi_b - \psi_b \frac{\partial}{\partial z}\psi_a\right]_{z=0}$

I_J is the so-called Josephson amplitude, and one can easily see that it's a positive quantity.

* From our discussion on hydrodynamics we know that the phase obeys $\hbar \dfrac{\partial S}{\partial t} = -\left[\dfrac{m}{2} v_s^2 + \mu\right] \underset{\substack{\uparrow \\ v_s \text{ very small} \\ \text{(barrier very high)}}}{\simeq} -\mu$

Then $\hbar \dfrac{\partial}{\partial t}(S_a - S_b) = -(\mu_a - \mu_b) \longrightarrow \dfrac{\partial \phi}{\partial t} = -\dfrac{1}{\hbar}(\mu_a - \mu_b)$

* Let $k = \dfrac{1}{2}(N_a - N_b) \ll \dfrac{N}{2}$ the number deviation from equilibrium ($N_a = N/2 + k$, $N_b = N/2 - k$)

Then $\mu_a = \mu_a(N_a) \overset{\text{Taylor}}{\simeq} \mu_a\left(\dfrac{N}{2}\right) + \left(\dfrac{\partial \mu_a}{\partial N_a}\right)_{N/2} k$

Hence $\dfrac{\partial \phi}{\partial t} \simeq -\dfrac{1}{\hbar}\left(\dfrac{\partial \mu}{\partial N}\right)_{N/2} 2k = -\dfrac{E_C}{\hbar} k$ with $E_C = 2\left(\dfrac{\partial \mu}{\partial N}\right)_{N/2}$

* In addition;

we know that $\dfrac{\partial N_b}{\partial t} = -I_J \sin\phi$, since $N_b = \dfrac{N}{2} - k \longrightarrow \dfrac{\partial k}{\partial t} = +I_J \sin\phi$

. We have hence 2 equations for k and ϕ:

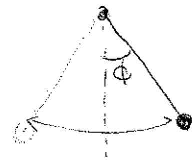
$$\boxed{\begin{aligned} \frac{\partial \phi}{\partial t} &= -\frac{E_C}{\hbar} k \\ \frac{\partial k}{\partial t} &= +\frac{E_J}{\hbar} \sin\phi \end{aligned}}$$

where we have introduced the so-called Josephson energy $E_J = \hbar I_J$

* Note that $E_C = 0$ in an ideal gas (since for ideal gases $\dfrac{d\mu}{dN} = 0$) whereas $E_C = \dfrac{2}{5}\dfrac{\mu}{N}$ in the Thomas-Fermi regime (because $\mu_{TF} \sim N^{2/5}$) p. 118

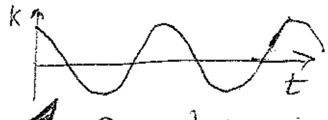
(173)

* Note that the eqs for ϕ and k are exactly those for a pendulum. As for a pendulum the solutions have different forms depending on the initial conditions



- If both k and ϕ are small $\rightarrow \sin\phi \simeq \phi$

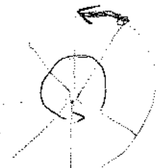
and $\frac{\partial k}{\partial t} \simeq -\frac{E_J}{\hbar}\phi \longrightarrow \frac{\partial^2 k}{\partial t^2} = -\frac{E_J E_C}{\hbar^2} k = -\omega_{pl}^2 k$



where $\omega_{pl} = \frac{\sqrt{E_J E_C}}{\hbar}$ is the plasma frequency

Hence the population oscillates between the two wells (Josephson oscillations)

- If $k > \sqrt{\frac{2E_J}{E_C}}$ the solutions of the pendulum eqs. correspond to a full rotation of the phase



- For an initial $k = k_0 >> E_J/E_C$, k is almost a constant.

This is easy to see: Formally we can solve the pendulum eqs. in the form:

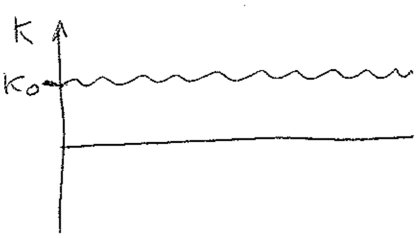
$$k(\tau) = k_0 + \int_0^{\tau} d\tau' \sin\left[\phi_0 - \frac{E_C}{E_J}\int_0^{\tau'} k(\tau'')\,d\tau''\right] \qquad \left(\tau = \frac{E_J}{\hbar}t\right)$$

If $k_0 >> \frac{E_J}{E_C} \rightarrow$ the argument of the sinus oscillates very fast and averages to a small quantity.

Hence $k(\tau) \simeq k_0$. The 1st correction is (as one can easily see from the previous equation)

$$k(\tau) = k_0 + \frac{E_J}{E_C k_0}\cos\left(\frac{E_C k_0}{E_J}\tau\right)$$

Hence there are oscillations (of very small amplitude) around k_0:



i.e. in spite of a non-zero tunneling, the population gets trapped in one of the wells. This is the so-called self-trapping

* We may write the pendulum eqs. in the convenient form:

$$\left.\begin{aligned}\frac{\partial \Phi}{\partial t} &= \frac{\partial}{\partial(\hbar k)} H_J \\ \frac{\partial(\hbar k)}{\partial t} &= -\frac{\partial}{\partial \Phi} H_J\end{aligned}\right\} \text{ with } \boxed{H_J = \frac{E_C k^2}{2} - E_J \cos\Phi}$$

Hence Φ and $\hbar k$ are canonically conjugated variables, and H_J is the corresponding Hamiltonian → Josephson Hamiltonian

*(Note: Actually $E_J = \frac{\hbar^2}{m}\sqrt{N_a N_b}\int dx \int dy \left[\phi_a \frac{\partial \phi_b}{\partial z} - \phi_b \frac{\partial \phi_a}{\partial z}\right]_{z=0}$

where we have used $\psi_{a,b} = \sqrt{N_{a,b}}\,\phi_{a,b}$, and $\phi_{a,b}$ are normalized to 1.

Hence $E_J = \delta_J \sqrt{N_a N_b} = \frac{\delta_J}{2}\sqrt{N^2 - 4k^2}$, where $\delta_J = \frac{\hbar^2}{m}\int dx dy \left(\phi_a \frac{\partial \phi_b}{\partial z} - \phi_b \frac{\partial \phi_a}{\partial z}\right)_{z=0}$.

In principle δ_J depends also on $N_{a,b}$ but mildly on k, if $k \ll N/2$.

Then $H_J = E_C \frac{k^2}{2} - \frac{\delta_J}{2}\sqrt{N^2 - 4k^2}\cos\phi$.

Note that if $E_C \gg \delta/N$, we may neglect this correction and use H_J as indicated at the top of this page.)

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* <u>JOSEPHSON EFFECT</u> (<u>BEYOND MEAN-FIELD</u>)

• As mentioned above ϕ and $\hbar k$ are canonically conjugated variables. We may hence easily quantize

$$\left.\begin{aligned}\phi &\longrightarrow \hat\phi \\ \hbar k &\longrightarrow \hbar \hat k\end{aligned}\right\} \; [\hat\phi, \hbar \hat k] = i\hbar \longrightarrow [\hat\phi, \hat k] = i \longrightarrow \hat k = -i\frac{\partial}{\partial \phi}$$

(ϕ-representation)

Hence $\boxed{\hat H_J = -\frac{E_C}{2}\frac{\partial^2}{\partial \phi^2} - E_J \cos\phi}$

• Due to the periodicity constraint, the uncertainty relation obeyed by the fluctuations of $\hat\phi$ and $\hat k$ takes the form:

$$(\Delta K)^2(\Delta(\sin(\phi-\phi_0)))^2 \geq \left[\frac{i}{2}\langle[K, \sin(\phi-\phi_0)]\rangle\right]^2 = \frac{1}{4}\langle\cos(\phi-\phi_0)\rangle^2 \quad (175)$$

where $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$). This holds for all values of Φ_0

* Only if the phase is localized around ϕ, we may then approximate $(\Delta K)^2(\Delta\phi)^2 \geq 1/4$ (i.e. the usual form for the uncertainty)

* The quantity

$$\alpha \equiv \langle\cos(\phi-\phi_0)\rangle$$

is the so-called coherence factor and is physically very important, since it provides the degree of phase coherence of the system.

If Φ is localized around $\Phi_0 \longrightarrow \alpha = 1 \longrightarrow$ full coherence.

On the contrary, if the relative phase is fully random $\rightarrow \alpha = 0$
$\longrightarrow$ absence of coherence.

* The coherence factor is directly related with the visibility of fringes in interference patterns.

* Let's see first what happens in the strong-tunneling limit $\frac{E_C}{E_J} \ll 1$.
In this case ϕ makes small-amplitude oscillations around $\phi = 0$ and we can expand

$$\hat{H}_J \simeq -\frac{E_C}{2}\frac{\partial^2}{\partial\phi^2} + \frac{E_J\phi^2}{2}$$

(we have neglected an unimportant constant)

As we already knew for small ϕ the system behaves like an harmonic oscillator with frequency $\omega_{pl} = \sqrt{E_C E_J}/\hbar$. The ground state is hence $\psi(\Phi) = \frac{1}{\sqrt{\sqrt{\pi}\, l_{HO}}} e^{-\phi^2/2l_{HO}^2}$, $l_{HO} = \left(\frac{E_C}{E_J}\right)^{1/4}$

Hence $(\Delta\phi)^2 = \langle\phi^2\rangle = \int d\phi\, \phi^2 \psi(\phi) = \frac{1}{2}\sqrt{\frac{E_C}{E_J}} \ll 1$

Phase fluctuations are (as expected) small. On the contrary, since in the ground state, then for $\phi \simeq 0$ we have $(\Delta K)^2(\Delta\phi)^2 = 1/4$ (it's an harmonic oscillator!)

$$(\Delta k)^2 = \frac{1}{2}\left(\frac{E_J}{E_C}\right)^{1/2} \gg 1 \longrightarrow \text{large relative number fluctuation}$$

* Let's see now what happens in the opposite case of weak tunneling $\frac{E_C}{E_J} \gg 1$. Then we may approximate

$$\hat{H}_J \simeq -\frac{E_C}{2}\frac{\partial^2}{\partial \Phi^2}$$

The eigenstates are plane waves $e^{in\Phi}$ with $n \in \mathbb{Z}$. $(n \neq 0)$
Hence the ground state in the Φ representation is a constant! Therefore the relative phase is completely arbitrary, being distributed in a random way.

* From the many-body point of view the ground state for $\frac{E_C}{E_J} \gg 1$ corresponds to a Fock state $|k=0\rangle$ (p. 168).

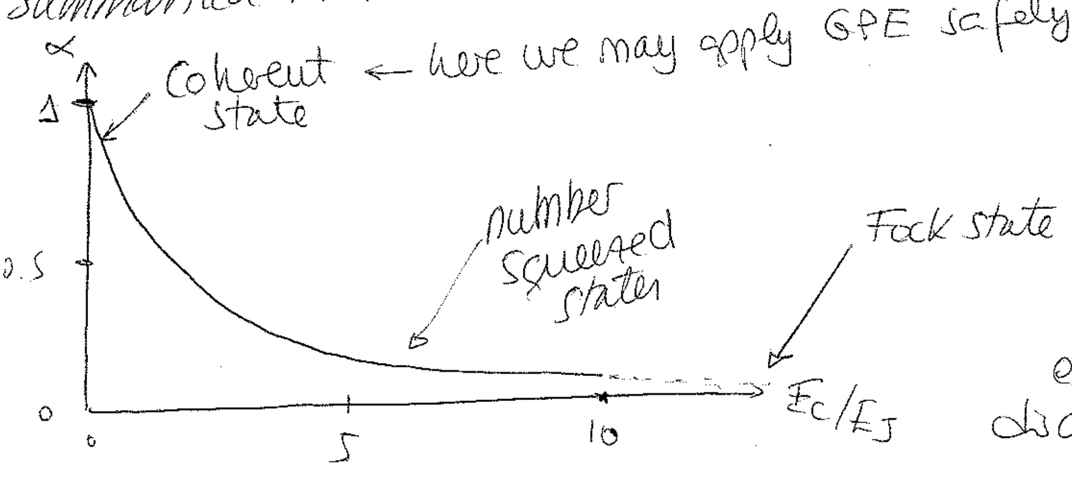
Introducing the first corrections due to the $-E_J \cos\phi$ term one may find that

$$\alpha \simeq 2\frac{E_J}{E_C} \longrightarrow \text{coherence vanishes as } E_J$$

$$(\Delta k)^2 \simeq 2\left(\frac{E_J}{E_C}\right)^2 \longrightarrow \text{number fluctuations vanish as } E_J^2$$

(Note: for a fixed E_J/E_C coherence may be also lost by thermal phase fluctuations. This may be employed for thermometry as recently shown at M. Oberthaler's group)

* The behaviour of the system as a function of E_C/E_J may be summarized in the following figure for $\alpha(E_C/E_J)$



* So when the tunneling decreases the GP formalism results inadequate, because the system isn't phase coherent. This will be especially important in our discussion on optical lattices.

* We may see this point from another perspective based in the idea of <u>quantum depletion of the condensate</u>.

- Let's consider the 2nd quantized Hamiltonian

$$\hat{H} = \int d^3r\, \hat{\psi}^\dagger(\vec{r})\left(\frac{-\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r})\right)\hat{\psi}(\vec{r}) + \frac{g}{2}\int d^3r\, \hat{\psi}^\dagger(\vec{r})\hat{\psi}^\dagger(\vec{r})\hat{\psi}(\vec{r})\hat{\psi}(\vec{r})$$

where $V_{ext}(\vec{r})$ is the double-well potential.

- We employ the ansatz:

$$\hat{\psi}(\vec{r}) = \underbrace{\phi_a(\vec{r})\,\hat{a}}_{\text{left well}} + \underbrace{\phi_b(\vec{r})\,\hat{b}}_{\text{right well}}$$

$\phi_{a,b}(\vec{r})$ are the ground-state single-particle wave-functions for the two separate wells.

Note that in the quasi-ideal regime, due to tunneling the ground state of the two well potential is given by the <u>symmetric combination</u> $\phi_0 = (\phi_a + \phi_b)/\sqrt{2}$, and the excited state will be $\phi_1 = (\phi_a - \phi_b)/\sqrt{2}$.

* <u>In the tight-binding approximation</u> (almost full separation of the wells)

$$\boxed{\hat{H}_{BH} \simeq \frac{E_C}{4}\left[\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}\right] - \frac{\delta_J}{2}\left(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\right)}$$

with $E_C = 2g\int \phi_a^4\, d^3r$

$$\delta_J = -2\int d^3r\, \phi_a(\vec{r})\left(\frac{-\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r})\right)\phi_b(\vec{r})$$

The previous Hamiltonian $\hat{H}_{BH}$ is the so-called <u>Bose-Hubbard Hamiltonian</u> (we will find it again in our future discussion on optical lattices)

* let $\hat{a}_0 = \frac{1}{\sqrt{2}}(\hat{a}+\hat{b})$

$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{a}-\hat{b})$

We employ Bogoliubov approximation (assuming BEC into ϕ_0) (p. 100)

$\hat{a}_0 \simeq \sqrt{N}$, $\hat{a}_0^+ \simeq \sqrt{N}$ } Now we proceed very similarly as in p. 101)

Retaining terms quadratic is $\hat{a}_1, \hat{a}_1^+$ we obtain

$$\hat{H} \simeq \delta_J \hat{a}_1^+\hat{a}_1 + \frac{E_C}{8} N \left(\hat{a}_1^+\hat{a}_1^+ + \hat{a}_1\hat{a}_1 + 2\hat{a}_1^+\hat{a}_1\right)$$

We employ the Bogoliubov transformation

$$\left.\begin{array}{l}\hat{a}_1 = u\hat{\alpha} + v\hat{\alpha}^+ \\ \hat{a}_1^+ = v\hat{\alpha} + u\hat{\alpha}^+\end{array}\right\} \quad \text{with} \quad u, v = \pm\sqrt{\frac{\delta_J + NE_C/4}{2\epsilon_J} \pm \frac{1}{2}}$$

$$\epsilon_J = \sqrt{\delta_J\left(\delta_J + \frac{NE_C}{2}\right)}$$

✓ Equivalent of the Bogoliubov spectrum

Then $\hat{H} \simeq \text{const} + \epsilon_J \hat{\alpha}^+\hat{\alpha}$

* The important point for our discussion here is the quantum depletion of the condensate (p. 105)

$$\delta N_0 = v^2 = \frac{\delta_J + NE_C/4}{\epsilon_J} - 1/2$$

(Note: since $N_0 + N_1 = N$, the δN_0 coincides with the population in ϕ_1)

Note that the applicability of the Bogoliubov approximation demands $\delta N_0 \ll N \longrightarrow E_C \ll 2E_J$.

* Hence the GP-formalism is only valid when $E_C/E_J \ll 1$ (the coherent regime of p. 176). When $E_C/E_J \gtrsim 1$, fragmentation occurs (i.e. both ϕ_0 and ϕ_1 are macroscopically occupied) and we can't use the simple GP-formalism any more.